SINGULARITIES AND ANALYTIC TORSION

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ABSTRACT. We prove the logarithmic divergence of equivariant analytic torsion for one-parameter degenerations of projective algebraic manifolds, when the coefficient vector bundle is given by a Nakano semi-positive vector bundle twisted by the relative canonical bundle.

CONTENTS

1. INTRODUCTION

Let $\mathcal{X}$ be a connected projective algebraic manifold of dimension $n+1$ and let C be a compact Riemann surface. Let $\pi\colon \mathcal{X} \to C$ be a surjective holomorphic map with critical locus Σ_π and with connected fibers.

Let G be a compact Lie group acting holomorphically on $\mathcal{X}$ and preserving the fibers of π. Assume that there exists a G-equivariant ample line bundle on $\mathcal{X}$. Set $\Delta = \pi(\Sigma_\pi)$, $C^o = C \setminus \Delta$, $\mathcal{X}^o = \mathcal{X}|_{\pi^{-1}(C^o)}$, $\pi^o = \pi|_{\mathcal{X}^o}$ and $X_s := \pi^{-1}(s)$ for $s \in C$. Then $\pi^o\colon \mathcal{X}^o \to C^o$ is a family of projective algebraic manifolds with G-action.

Let $T\mathcal{X}/C$ be the G-equivariant subbundle of $T\mathcal{X}|_{\mathcal{X}\setminus\Sigma_\pi}$ defined as $T\mathcal{X}/C = \ker \pi_*|_{\mathcal{X}\setminus\Sigma_\pi}$. Let $h_{\mathcal{X}}$ be a G-invariant Kähler metric on $\mathcal{X}$ and set $h_{\mathcal{X}/C} = h_{\mathcal{X}}|_{T\mathcal{X}/C}$. Let $\omega_{\mathcal{X}} = \Omega_{\mathcal{X}}^{n+1}$ be the canonical bundle of $\mathcal{X}$ and let $\omega_{\mathcal{X}/C} = \Omega_{\mathcal{X}}^{n+1} \otimes (\pi^*\Omega_C^1)^{-1}$ be the relative canonical bundle of $\pi\colon \mathcal{X} \to C$. Let $\xi \to \mathcal{X}$ be a G-equivariant holomorphic vector bundle on $\mathcal{X}$ equipped with a G-invariant Hermitian metric h_ξ. We write $\omega_{\mathcal{X}/C}(\xi) = \omega_{\mathcal{X}/C} \otimes \xi$. We set $\xi_s = \xi|_{X_s}$ for $s \in C$.

Let $0 \in \Delta$ be a critical value of π. Let (S, s) be a coordinate neighborhood of C centered at 0 such that $S \cap \Delta = \{0\}$. We set $X = \pi^{-1}(S)$ and $S^o = S \setminus \{0\}$.

For $g \in G$ and $s \in S^o$, let $\tau_G(X_s, \omega_{X_s}(\xi_s))(g)$ be the equivariant analytic torsion [6] of $(X_s, \omega_{X_s}(\xi_s))$ with respect to $h_{X_s} = h_{\mathcal{X}}|_{X_s}$ and $h_{\xi_s} = h_\xi|_{X_s}$, where $\omega_{X_s} = \Omega^n_{X_s}$ for $s \neq 0$. The goal of this article is to determine the behavior of $\tau_G(X_s, \omega_{X_s}(\xi_s))(g)$ as $s \to 0$, when (ξ, h_ξ) is *Nakano semi-positive* on X. (See Sect.5.1 for the notion of Nakano semi-positivity.) Notice that all $R^q \pi_* \omega_{X/S}(\xi)$ are locally free in this case by Takegoshi's torsion freeness theorem [28] and the condition $\dim S = 1$. To express the singularity of $\tau_G(X_s, \omega_{X_s}(\xi_s))(g)$ as $s \to 0$ in more detail, we briefly recall Gauss maps and semistable reductions.

Let $\mathbf{P}(T\mathcal{X})^\vee$ be the G-equivariant projective-space bundle such that $\mathbf{P}(T\mathcal{X})^\vee_x = \mathbf{P}(T_x\mathcal{X})^\vee$ is the set of n-dimensional linear subspaces of $T_x\mathcal{X}$ for $x \in \mathcal{X}$. The Gauss map $\gamma \colon \mathcal{X} \setminus \Sigma_\pi \to \mathbf{P}(T\mathcal{X})^\vee$ is the section defined as $\gamma(x) = \ker(\pi_*)_x \in \mathbf{P}(T_x\mathcal{X})^\vee$ for $x \in \mathcal{X} \setminus \Sigma_\pi$. Since γ extends to a rational map $\gamma \colon \mathcal{X} \dashrightarrow \mathbf{P}(T\mathcal{X})^\vee$, there is a resolution $q \colon (\widetilde{\mathcal{X}}, E) \to (\mathcal{X}, \Sigma_\pi)$ of the indeterminacy of γ with $q|_{\widetilde{\mathcal{X}} \setminus E} \colon \widetilde{\mathcal{X}} \setminus E \cong \mathcal{X} \setminus \Sigma_\pi$ such that $\widetilde{\gamma} = \gamma \circ q$ extends to a holomorphic map from $\widetilde{\mathcal{X}}$ to $\mathbf{P}(T\mathcal{X})^\vee$ and such that E is a normal crossing divisor of $\widetilde{\mathcal{X}}$. Since γ is G-equivariant, we may assume that G acts on $\widetilde{\mathcal{X}}$ and that q and $\widetilde{\gamma}$ are G-equivariant [4]. We denote by $\mathcal{H} = \mathcal{O}_{\mathbf{P}(T\mathcal{X})^\vee}(1)$ the tautological quotient bundle on $\mathbf{P}(T\mathcal{X})^\vee$.

For $g \in G$, let $\mathcal{X}^g = \{x \in \mathcal{X}; g \cdot x = x\}$ be its fixed-point set. Since g is an isometry of $\mathcal{X}$, $\mathcal{X}^g$ is the disjoint union of compact complex submanifolds of $\mathcal{X}$:

$$\mathcal{X}^g = \mathcal{X}^g_H \amalg \mathcal{X}^g_V.$$

Here $\mathcal{X}^g_H$ is a horizontal submanifold, i.e., $\pi|_{\mathcal{X}^g_H} \colon \mathcal{X}^g_H \to C$ is a flat holomorphic map and $\mathcal{X}^g_V$ is a vertical submanifold, i.e., $\pi(\mathcal{X}^g_V)$ is a proper subset of C. Since the G-action on C is trivial, one has $\mathcal{X}^g_V \subset \Sigma_\pi$ and $\pi(\mathcal{X}^g_V) \subset \Delta$ by the G-equivariance of π. Let $\widetilde{\mathcal{X}}^g_H \subset \widetilde{\mathcal{X}}$ be the proper transform of $\mathcal{X}^g_H \subset \mathcal{X}$. Since $\widetilde{\mathcal{X}}^g_H \subset (\widetilde{\mathcal{X}})^g$, we get $\widetilde{\gamma}(\widetilde{\mathcal{X}}^g_H) \subset (\mathbf{P}(T\mathcal{X})^\vee)^g$ by the G-equivariance of $\widetilde{\gamma}$. Hence $g \in G$ preserves the fibers of $(\widetilde{\gamma}^* \mathcal{U})|_{\widetilde{\mathcal{X}}^g_H}$. We set $E_0 = (\pi \circ q)^{-1}(0) \cap E$ and define

$$\alpha_g(X_0, \omega_{X/S}(\xi)) = \int_{E_0 \cap \widetilde{\mathcal{X}}^g_H} \widetilde{\gamma}^* \left\{ \frac{\mathrm{Td}(\mathcal{H}^\vee)^{-1} - 1}{c_1(\mathcal{H}^\vee)} \right\} q^* \{\mathrm{Td}_g(T\mathcal{X}) \mathrm{ch}_g(\omega_{\mathcal{X}}(\xi))\}$$

$$- \int_{\mathcal{X}^g_V \cap X_0} \mathrm{Td}_g(T\mathcal{X}) \mathrm{ch}_g(\omega_{\mathcal{X}}(\xi)).$$

Let $f \colon (Y, Y_0) \to (T, 0)$ be a semistable reduction of $\pi \colon (X, X_0) \to (S, 0)$. We have a commutative diagram, where $Y_0 \subset Y$ is a reduced normal crossing divisor:

$$
\begin{array}{ccc}
(Y, Y_0 = f^{-1}(0)) & \xrightarrow{\ F\ } & (X, X_0) \\
f \downarrow & & \pi \downarrow \\
(T, 0) & \xrightarrow{\ \mu\ } & (S, 0).
\end{array}
$$

By [23], $R^q f_* \omega_{Y/T}(F^*\xi)$ is a G-equivariant locally free sheaf equipped with an injective homomorphism $\varphi \colon R^q f_* \omega_{Y/T}(F^*\xi) \to \mu^* R^q \pi_* \omega_{X/S}(\xi)$ of G-modules. We regard $R^q f_* \omega_{Y/T}(F^*\xi)$ as a subsheaf of $\mu^* R^q \pi_* \omega_{X/S}(\xi)$ by this inclusion. The Lefschetz trace of the G-action on $(\mu^* R \pi_* \omega_{X/S}(\xi)/R f_* \omega_{Y/T}(F^*\xi))_0$ is defined as

$$\mathbb{L}_g \left(\frac{\mu^* R \pi_* \omega_{X/S}(\xi)}{R f_* \omega_{Y/T}(F^*\xi)} \right) = \sum_{q \geq 0} (-1)^q \mathrm{Tr} \left[g|_{(\mu^* R^q \pi_* \omega_{X/S}(\xi)/R^q f_* \omega_{Y/T}(F^*\xi))_0} \right]$$

for $g \in G$. Now the main result of this article is stated as follows.

Theorem 1.1. *If (ξ, h_ξ) is Nakano semi-positive on $X = \pi^{-1}(S)$, then there exist constants $\nu_g, c_g \in \mathbf{C}$ such that as $s \to 0$*

$$\log \tau_G(X_s, \omega_{X_s}(\xi_s))(g) = \{\alpha_g(X_0, \omega_{X/S}(\xi)) + \frac{1}{\deg \mu} \mathbb{L}_g \left(\frac{\mu^* R\pi_* \omega_{X/S}(\xi)}{Rf_* \omega_{Y/T}(F^*\xi)} \right)\} \log |s|^2$$
$$+ \nu_g \, \log(-\log |s|^2) + c_g + O\left(1/\log |s|\right).$$

By Theorem 1.1, the logarithmic singularity of $\log \tau_G(X_s, \omega_{X_s}(\xi_s))(g)$ is determined by the algebraic term $\mathbb{L}_g(\mu^* R\pi_* \omega_{X/S}(\xi)/Rf_* \omega_{Y/T}(F^*\xi))$ measuring the cohomological difference between (X, X_0) and its semistable reduction (Y, Y_0) and the topological term $\alpha_g(X_0, \omega_{X/S}(\xi))$ arising from the resolution of the Gauss map. See Corollary 6.10 for a formula for $\log \tau_G(X_s, \xi_s)(g)$ as $s \to 0$, when (ξ, h_ξ) is *semi-negative in the dual Nakano sense*. We remark $\nu_g \in \mathbf{Z}$ if $g = 1$.

For some important classes of degenerations, we get the following.

Theorem 1.2. *Assume that (ξ, h_ξ) is Nakano semi-positive on $X = \pi^{-1}(S)$.*
(1) *If X_0 is reduced and the pair $(\mathcal{X}, X_0)$ has only log-canonical singularities, then $\mathbb{L}_g(\mu^* R\pi_* \omega_{X/S}(\xi)/Rf_* \omega_{Y/T}(F^*\xi)) = 0$ in Theorem 1.1.*
(2) *If X_0 is reduced, normal and has only canonical (equivalently rational) singularities, then there exist $c \in \mathbf{C}$, $r \in \mathbf{Q}_{>0}$, $l \in \mathbf{Z}_{\geq 0}$ such that as $s \to 0$,*

$$\log \tau_G(X_s, \omega_{X_s}(\xi_s))(g) = \alpha_g(X_0, \omega_{X/S}(\xi)) \log |s|^2 + c + O\left(|s|^r (\log |s|)^l\right).$$

For the definition of (log-)canonical singularities, see Sect.7.1. Since the pair $(\mathcal{X}, X_0)$ has only log-canonical singularities when $\pi \colon (X, X_0) \to (S, 0)$ is semistable, Theorem 1.1 and Theorem 1.2 (1) are compatible. If the singularity of $(\mathcal{X}, X_0)$ is strictly worse than log-canonical, then $\mathbb{L}_g(\mu^* R\pi_* \omega_{X/S}(\xi)/Rf_* \omega_{Y/T}(F^*\xi)) \neq 0$ in general (Sect. 8). When $\dim X_s = 1$, $G = \{1\}$ and X_0 has at most nodes, asymptotic expansions like Theorems 1.1 and 1.2 (1) were obtained by Bismut–Bost [8] and Wolpert [29], where $\nu_g \neq 0$ in general. Without the assumption of the curvature of (ξ, h_ξ), the singularity of analytic torsion was determined by Farber [14] when X_0 is *non-singular*. In [33], we use Theorem 1.1 to prove the automorphic property of the invariant τ_M introduced in [30], which plays a crucial role to determine an explicit Borcherds product expressing τ_M in the case $r(M) \geq 18$ (cf. [32]).

The strategy for the proof of Theorem 1.1 is as follows. Following [7, Th. 5.9], [31, Th. 1.1], we determine the singularity of the equivariant Quillen metric on the equivariant determinant of the cohomologies of ξ by applying the Bismut immersion formula [6] to the G-equivariant embedding $X_s \hookrightarrow \mathcal{X}$ (Theorem 4.1). Then we get Theorem 1.1 by studying the behavior of the L^2-metric on $R^q \pi_* \omega_{X/S}(\xi)$ (Theorem 6.8). For this, a theorem of Takegoshi [28] and its extension by Mourougane-Takayama [23] play a crucial role to express the fiberwise harmonic representative of an element of $H^q(X, \Omega_X^{n+1}(\xi))$, where the Nakano semi-positivity of ξ is essentially used (Sect. 6). The asymptotic expansion follows from a theorem of Barlet [2].

This article is organized as follows. In Sect.2, we recall Gauss maps. In Sect.3, we recall equivariant Quillen metrics and study their regularity. In Sect.4, we determine the singularity of equivariant Quillen metrics. In Sect.5, we recall the notion of Nakano semi-positivity of vector bundles. In Sect.6, we prove Theorem 1.1. In Sect.7, we prove Theorem 1.2. In Sect.8, we study some examples. In Sect.9, we prove some technical results.

Notation For a complex manifold, we set $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$. Hence $dd^c = \frac{1}{2\pi i}\bar{\partial}\partial$. For a complex manifold Y, $A_Y^{p,q}$ denotes the vector space of C^∞ (p,q)-forms on Y. We set $\widetilde{A}_Y = \bigoplus_{p\geq 0} A_Y^{p,p}/\operatorname{Im}\partial + \operatorname{Im}\bar{\partial}$. For a G-equivariant vector bundle F over Y equipped with a G-invariant Hermitian metric h_F, we denote by $c_i(F, h_F) \in \bigoplus_{p\geq 0} A_Y^{p,p}$, $\operatorname{Td}_g(F, h_F), \operatorname{ch}_g(F, h_F) \in \bigoplus_{p\geq 0} A_{Y^g}^{p,p}$ the i-th Chern form, the equivariant Todd form, and the equivariant Chern character form of (F, h_F) with respect to the holomorphic Hermitian connection, respectively (cf. [6]).

After [2], we set $\mathcal{B}(S) = C^\infty(S) \oplus \bigoplus_{r\in\mathbf{Q}\cap(0,1]}\bigoplus_{k=0}^n |s|^{2r}(\log|s|)^k C^\infty(S)$. A function of $\mathcal{B}(S)$ is continuous and has an asymptotic expansion at $s = 0$. We write $\phi \equiv_\mathcal{B} \psi$ if $\phi, \psi \in C^\infty(S^o)$ satisfies $\phi - \psi \in \mathcal{B}(S) \subset C^0(S)$. Throughout this article, we keep the notation and the assumptions in Sect.1.

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2. The Gauss map and its equivariant resolution

Let $\Omega_\mathcal{X}^1$ be the holomorphic cotangent bundle of $\mathcal{X}$. Let $\Pi\colon \mathbf{P}(\Omega_\mathcal{X}^1\otimes\pi^*TC) \to \mathcal{X}$ be the projective-space bundle associated with $\Omega_\mathcal{X}^1\otimes\pi^*TC$. Let $\Pi^\vee\colon \mathbf{P}(T\mathcal{X})^\vee \to \mathcal{X}$ be the dual projective-space bundle of $\mathbf{P}(T\mathcal{X})$, whose fiber $\mathbf{P}(T_x\mathcal{X})^\vee$ is the set of hyperplanes of $T_x\mathcal{X}$ passing through $0_x \in T_x\mathcal{X}$. Since $\dim C = 1$, we have $\mathbf{P}(\Omega_\mathcal{X}^1 \otimes \pi^*TC) = \mathbf{P}(\Omega_\mathcal{X}^1) \cong \mathbf{P}(T\mathcal{X})^\vee$.

Let $x \in \mathcal{X}\setminus\Sigma_\pi$. Let s be a holomorphic local coordinate of C near $\pi(x) \in C$. We define the Gauss maps $\nu\colon \mathcal{X}\setminus\Sigma_\pi \to \mathbf{P}(\Omega_\mathcal{X}^1\otimes\pi^*TC)$ and $\gamma\colon \mathcal{X}\setminus\Sigma_\pi \to \mathbf{P}(T\mathcal{X})^\vee$ by

$$\nu(x) := [d\pi_x] = \left[\sum_{i=0}^n \frac{\partial(s\circ\pi)}{\partial z_i}(x)\,dz_i\otimes\frac{\partial}{\partial s}\right], \qquad \gamma(x) := [T_x\mathcal{X}_{\pi(x)}].$$

Under the canonical isomorphism $\mathbf{P}(\Omega_\mathcal{X}^1 \otimes \pi^*TC) \cong \mathbf{P}(T\mathcal{X})^\vee$, one has $\nu = \gamma$.

Let $\mathcal{H} := \mathcal{O}_{\mathbf{P}(T\mathcal{X})^\vee}(1)$ be the tautological quotient bundle and let $\mathcal{U}$ be the universal hyperplane bundle of $\mathbf{P}(T\mathcal{X})^\vee$. We have the exact sequence of G-equivariant vector bundles on $\mathbf{P}(T\mathcal{X})^\vee$

$$\mathcal{S}^\vee\colon 0 \longrightarrow \mathcal{U} \longrightarrow (\Pi^\vee)^*T\mathcal{X} \longrightarrow \mathcal{H} \longrightarrow 0.$$

Let $h_\mathcal{U}$ be the Hermitian metric on $\mathcal{U}$ induced from $(\Pi^\vee)^*h_\mathcal{X}$, and let $h_\mathcal{H}$ be the Hermitian metric on $\mathcal{H}$ induced from $(\Pi^\vee)^*h_\mathcal{X}$ by the C^∞-isomorphism $\mathcal{H}\cong\mathcal{U}^\perp$. On $\mathcal{X}\setminus\Sigma_\pi$, we have $(T\mathcal{X}/C, h_{\mathcal{X}/C}) = \gamma^*(\mathcal{U}, h_\mathcal{U})$.

Let $\mathcal{L} := \mathcal{O}_{\mathbf{P}(\Omega_\mathcal{X}^1\otimes\pi^*TC)}(-1) \subset \Pi^*(\Omega_\mathcal{X}^1 \otimes \pi^*TC)$ be the tautological line bundle over $\mathbf{P}(\Omega_\mathcal{X}^1\otimes\pi^*TC)$. Let h_C be a Hermitian metric on C. Let $h_{\Omega_\mathcal{X}^1}$ be the Hermitian metric on $\Omega_\mathcal{X}^1$ induced from $h_\mathcal{X}$. Let $h_\mathcal{L}$ be the Hermitian metric on $\mathcal{L}$ induced from the metric $\Pi^*(h_{\Omega_\mathcal{X}^1}\otimes\pi^*h_C)$ by the inclusion $\mathcal{L} \subset \Pi^*(\Omega_\mathcal{X}^1\otimes\pi^*TC)$.

Since Σ_π is a proper subvariety of $\mathcal{X}$, the Gauss maps ν and γ extend to rational maps $\nu\colon \mathcal{X} \dashrightarrow \mathbf{P}(\Omega_\mathcal{X}^1\otimes\pi^*TC)$ and $\gamma\colon \mathcal{X} \dashrightarrow \mathbf{P}(T\mathcal{X})^\vee$. By [4, Th. 13.2], there exist a projective algebraic manifold $\widetilde{\mathcal{X}}$, a normal crossing divisor $E \subset \widetilde{\mathcal{X}}$, a birational holomorphic map $q\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ with $E = q^{-1}(\Sigma_\pi)$, and holomorphic maps $\widetilde{\nu}\colon \widetilde{\mathcal{X}} \to \mathbf{P}(\Omega_\mathcal{X}^1\otimes\pi^*TC)$ and $\widetilde{\gamma}\colon \widetilde{\mathcal{X}} \to \mathbf{P}(T\mathcal{X})^\vee$ with the following properties:

(a) The G-action on $\mathcal{X}$ lifts to a G-action on $\widetilde{\mathcal{X}}$ and $q|_{\widetilde{\mathcal{X}} \setminus E} \colon \widetilde{\mathcal{X}} \setminus E \to \mathcal{X} \setminus \Sigma_\pi$ is a G-equivariant isomorphism.

(b) $(\pi \circ q)^{-1}(b)$ is a normal crossing divisor of $\widetilde{\mathcal{X}}$ for all $b \in \Delta$.

(c) $\widetilde{\nu} = \nu \circ q$ and $\widetilde{\gamma} = \gamma \circ q$ on $\widetilde{\mathcal{X}} \setminus E$.

Then $\widetilde{\nu} = \widetilde{\gamma}$ under the canonical isomorphism $\mathbf{P}(\Omega^1_{\mathcal{X}} \otimes \pi^* TC) \cong \mathbf{P}(T\mathcal{X})^\vee$.

3. Equivariant Quillen metrics

In this section, we recall equivariant Quillen metrics and prove its smoothness for smooth projective morphisms. For more general treatments including smooth Kähler morphisms, we refer to [9, III Sects. 2 and 3]. In the rest of this article, $\widehat{G}$ denotes the set of equivalence classes of complex irreducible representations of G. For $W \in \widehat{G}$, the corresponding irreducible character is denoted by χ_W.

3.1. Equivariant analytic torsion and equivariant Quillen metrics.

3.1.1. *Equivariant analytic torsion.* Let V be a compact Kähler manifold with holomorphic G-action. Let h_V be a G-invariant Kähler metric on V. Let F be a G-equivariant holomorphic vector bundle on V equipped with a G-invariant Hermitian metric h_F. We set $\overline{V} = (V, h_V)$ and $\overline{F} = (F, h_F)$. Let $A^{p,q}_V(F)$ be the vector space of F-valued smooth (p, q)-forms on V. We set $S_F = \bigoplus_{q \geq 0} A^{0,q}_V(F)$, which is equipped with the L^2 metric $(\cdot, \cdot)_{L^2}$ with respect to h_V and h_F. Then $(\cdot, \cdot)_{L^2}$ is G-invariant with respect to the standard G-action on S_F.

Let $\Box_F = 2(\bar{\partial}_F + \bar{\partial}_F^*)^2$ be the Laplacian acting on S_F. We denote by $\sigma(\Box_F)$ the spectrum of $\Box_F$. Let $K_F(\lambda)$ be the eigenspace of $\Box_F$ with eigenvalue $\lambda \in \sigma(\Box_F)$. Since G preserves the metrics h_V and h_F, $\Box_F$ commutes with the G-action on S_F. Hence G acts on $K_F(\lambda)$. With respect to the $\mathbf{Z}$-grading on $K_F(\lambda)$ induced from the one on S_F, the number operator N and the supertrace $\mathrm{Tr}_s[\cdot]$ are defined on $K_F(\lambda)$ (cf. [9]). For $g \in G$ and $s \in \mathbf{C}$ with $\mathrm{Re}\, s \gg 0$, we define

$$\zeta_G(g)(s) := \sum_{\lambda \in \sigma(\Box_F) \setminus \{0\}} \lambda^{-s} \mathrm{Tr}_s \left[g\, N |_{K_F(\lambda)} \right].$$

Then $\zeta_G(g)(s)$ extends to a meromorphic function on $\mathbf{C}$ and is holomorphic at $s = 0$. For $g \in G$, we define

$$\log \tau_G(\overline{V}, \overline{F})(g) := -\zeta_G'(g)(0).$$

Then $\log \tau_G(\overline{V}, \overline{F})(g)$ is called the equivariant analytic torsion of $(\overline{V}, \overline{F})$ (cf. [6]).

3.1.2. *Equivariant Quillen metrics.* Since F is a G-equivariant holomorphic vector bundle on V, G acts on $H(V, F) = \bigoplus_{q \in \mathbf{Z}} H^q(V, F)$ and preserves its grading. One has the isotypical splitting of the $\mathbf{Z}$-graded vector space

$$H(V, F) = \bigoplus_{W \in \widehat{G}} \mathrm{Hom}_G(W, H(V, F)) \otimes W.$$

Since $\dim H(V, F) < +\infty$, $\mathrm{Hom}_G(W, H(V, F)) = 0$ except for finite $W \in \widehat{G}$. We set

$$\lambda_W(F) = \det \mathrm{Hom}_G(W, H(V, F)) \otimes W := \bigotimes_{q \geq 0} (\det \mathrm{Hom}_G(W, H^q(V, F)) \otimes W)^{(-1)^q}.$$

The equivariant determinant of the cohomologies of F is defined as

$$\lambda_G(F) := \prod_{W \in \widehat{G}} \lambda_W(F).$$

Notice that our sign convention is different form the one in [6, Eq. (2.9)]. When $\mathrm{Hom}_G(W, H(V, F)) = 0$, $\lambda_W(F)$ is canonically isomorphic to $\mathbf{C}$ by definition. In this case, the canonical element of $\lambda_W(F)$ corresponding to $1 \in \mathbf{C}$ is denoted by $1_{\lambda_W(F)}$. A vector $\alpha = (\alpha_W)_{W \in \widehat{G}} \in \lambda_G(F)$ is said to be *admissible* if $\alpha_W \neq 0$ for all $W \in \widehat{G}$ and if $\alpha_W = 1_{\lambda_W(F)}$ except for finitely many $W \in \widehat{G}$. The set of admissible elements of $\lambda_G(F)$ is identified with the direct sum $\bigoplus_{W \in \widehat{G}} \lambda_W(F)^\times$, where $\lambda_W(F)^\times := \lambda_W(F) \setminus \{0\}$ is the set of invertible elements of $\lambda_W(F)$.

By Hodge theory, we have an isomorphism of $\mathbf{Z}$-graded G-spaces $H(V, F) \cong K_F(0)$. The G-invariant metric on $H(V, F)$ induced from the L^2-metric on $K_F(0) \subset S_F$ by this isomorphism is denoted by $h_{H(V,F)}$. Then the isotypical splitting of $H(V, F)$ is orthogonal with respect to $h_{H(V,F)}$. Let $\| \cdot \|_{L^2, \lambda_W(F)}$ be the Hermitian metric on $\lambda_W(F)$ induced from $h_{H(V,F)}$. Recall that χ_W is the character of $W \in \widehat{G}$. For an admissible element $\alpha = (\alpha_W)_{W \in \widehat{G}} \in \bigoplus_{W \in \widehat{G}} \lambda_W(F)^\times$, we set

$$\log \|\alpha\|^2_{Q, \lambda_G(F)}(g) := -\zeta'_G(g)(0) + \sum_{W \in \widehat{G}} \frac{\chi_W(g)}{\dim W} \log \|\alpha_W\|^2_{L^2, \lambda_W(F)}.$$

The $\mathbf{C}$-valued function $\log \| \cdot \|^2_{Q, \lambda_G(F)}(g)$ on $\bigoplus_{W \in \widehat{G}} \lambda_W(F)^\times$ is called the equivariant Quillen metric on $\lambda_G(F)$ with respect to h_V, h_F. Notice that equivariant Quillen metric makes sense only for admissible elements. We refer to [25], [24], [6], [19], [21] for more about equivariant analytic torsion, equivariant determinants, and equivariant Quillen metrics.

3.2. The smoothness of equivariant Quillen metrics.
Let M be a compact complex manifold with holomorphic G-action and let B be a compact complex manifold with trivial G-action. Let $\pi \colon M \to B$ be a G-equivariant proper surjective flat holomorphic map. Assume that there is a G-equivariant ample line bundle on M. Let $F \to M$ be a G-equivariant holomorphic vector bundle. Let h_M be a G-invariant Kähler metric on M and let h_F be a G-invariant Hermitian metric on F. We set $M_b := \pi^{-1}(b)$ and $F_b := F|_{M_b}$ for $b \in B$. Then G preserves the fibers M_b and F_b, and $\pi \colon M \to B$ is a family of projective algebraic varieties.

For $W \in \widehat{G}$, we define $\mathrm{Hom}_G(W, R^q \pi_* F) \otimes W$ to be the sheaf on B associated to the presheaf $U \mapsto \mathrm{Hom}_G \left(W, H^q(\pi^{-1}(U), F|_{\pi^{-1}(U)}) \right) \otimes W$.

3.2.1. *Direct image sheaves.*
Since there is a G-equivariant ample line bundle on M by assumption, there exist a complex of G-equivariant holomorphic vector bundles $F_\bullet \colon 0 \longrightarrow F_0 \longrightarrow F_1 \longrightarrow \cdots \longrightarrow F_m \longrightarrow 0$ on M and a homomorphism $i \colon F \to F_0$ of G-modules with the following conditions:

  (i) The complex $0 \to F \to F_0 \to \cdots \to F_m \to 0$ is acyclic.
  (ii) $H^q(M_b, F_i|_{M_b}) = 0$ for all $q > 0$ and $i \geq 0$.

By (i), (ii), $\pi_* F_\bullet \colon 0 \longrightarrow \pi_* F_0 \longrightarrow \pi_* F_1 \longrightarrow \cdots \longrightarrow \pi_* F_m \longrightarrow 0$ is a complex of G-equivariant locally free sheaves of finite rank over B, whose cohomology sheaves

compute the direct image sheaves $R^q\pi_*F$, $q \geq 0$, i.e.,

(3.1)
$$R^q\pi_*F = \mathcal{H}^q(\pi_*F_\bullet) := \ker\{\pi_*F_q \to \pi_*F_{q+1}\}/\mathrm{Im}\{\pi_*F_{q-1} \to \pi_*F_q\}.$$

Since (3.1) is an equality of G-modules, we get for all $W \in \widehat{G}$ and $q \geq 0$

(3.2)
$$\mathrm{Hom}_G(W, R^q\pi_*F) \otimes W = \mathcal{H}^q(\mathrm{Hom}_G(W, \pi_*F_\bullet) \otimes W).$$

Since $\mathrm{Hom}_G(W, \pi_*F_\bullet) \otimes W$ is a complex of locally free $\mathcal{O}_B$-modules, we deduce from (3.2) that $\mathrm{Hom}_G(W, R^q\pi_*F) \otimes W$ is a coherent $\mathcal{O}_B$-module on B. Since $\mathcal{H}^q(\pi_*F_\bullet) = \bigoplus_{W \in \widehat{G}} \mathcal{H}^q(\mathrm{Hom}_G(W, \pi_*F_\bullet) \otimes W)$, we get the isotypical splitting on B

(3.3)
$$R^q\pi_*F = \bigoplus_{W \in \widehat{G}} \mathrm{Hom}_G(W, R^q\pi_*F) \otimes W.$$

Notice that $\mathrm{Hom}_G(W, R^q\pi_*F) = 0$ except for finitely many $W \in \widehat{G}$.

3.2.2. *Equivariant determinant of cohomologies.* Recall that for any coherent analytic sheaf $\mathcal{F}$ on B, one can associate the invertible sheaf $\det\mathcal{F}$ on B by [18], [9, III, Sect. 3]. We define

$$\begin{aligned}\lambda_W(F) &:= \bigotimes_{q \geq 0} \det\left(\mathrm{Hom}_G(W, R^q\pi_*F) \otimes W\right)^{(-1)^q}, \\ \lambda_G(F) &:= \prod_{W \in \widehat{G}} \lambda_W(F).\end{aligned}$$

If $\mathrm{Hom}_G(W, R^q\pi_*F) = 0$ for all $q \geq 0$, then $\lambda_W(F)$ is canonically isomorphic to $\mathcal{O}_B$. In this case, the canonical section of $\lambda_W(F)$ corresponding to $1 \in H^0(B, \mathcal{O}_B)$ is denoted by $1_{\lambda_W(F)}$. By [9, III, Lemma 3.7], there is a canonical identification

(3.4)
$$\lambda_W(F)_b := \lambda_W(F)/\mathfrak{m}_b\lambda_W(F) = \lambda_W(F|_{M_b})$$

for all $b \in B$, where $\mathfrak{m}_b$ is the maximal ideal of $\mathcal{O}_{B,b}$.

For an open subset $U \subset B$, a holomorphic section $\sigma = (\sigma_W)_{W \in \widehat{G}}$ of $\lambda_G(F)|_U$ is said to be *admissible* if σ_W is nowhere vanishing on U for all $W \in \widehat{G}$ and if $\sigma_W = 1_{\lambda_W(\eta)}$ except for finitely many $W \in \widehat{G}$.

3.2.3. *The smoothness of equivariant Quillen metrics.* Let $D \subset B$ be the discriminant locus of π and set $B^o := B \setminus D$. By (3.4), $\lambda_G(F)|_{B^o}$ is equipped with the equivariant Quillen metric $\|\cdot\|_{Q,\lambda_G(F)}(g)$ with respect to $h_M|_{TM/B}$ and h_F such that for every $b \in B^o$,

$$\|\cdot\|_{Q,\lambda_G(F)}(g)(b) := \|\cdot\|_{Q,\lambda_G(F_b)}(g).$$

Theorem 3.1. *Let $U \subset B^o$ be a small connected open subset with $\lambda_W(F_i)|_U \cong \mathcal{O}_U$ for all i and $W \in \widehat{G}$. Let $\sigma = (\sigma_W)_{W \in \widehat{G}}$ be an admissible holomorphic section of $\lambda_G(F)|_U$. Then $\log\|\sigma\|^2_{Q,\lambda_G(F)}(g) \in C^\infty(U)$.*

Proof. By the definition of $\lambda_W(F)$, we have the canonical isomorphism

$$\varphi_W : \lambda_W(F) = \bigotimes_{q \geq 0} \det\mathcal{H}^q(\mathrm{Hom}_G(W, \pi_*F_\bullet) \otimes W)^{(-1)^q} \cong \bigotimes_{i \geq 0} \lambda_W(F_i)^{(-1)^i}.$$

There exists an admissible holomorphic section $\sigma_i = ((\sigma_i)_W)_{W \in \widehat{G}}$ of $\lambda_G(F_i)|_U$ such that $\varphi_W(\sigma_W) = \otimes_{i \geq 0}(\sigma_i)_W^{(-1)^i}$ for all $W \in \widehat{G}$.

Let h_{F_i} be a G-invariant Hermitian metric on F_i and let $\|\cdot\|_{Q,\lambda_G(F_i)}(g)$ be the equivariant Quillen metric on $\lambda_G(F_i)$ with respect to the G-invariant metrics

$h_{M/B} = h_M|_{TM/B}$ and h_{F_i}. Let $\widetilde{\mathrm{ch}}_g(F, F_\bullet; h_F, h_{F_\bullet}) \in \widetilde{A}(M^g)$ be the Bott–Chern secondary class [9, I e), f)] such that

$$dd^c \widetilde{\mathrm{ch}}_g(F, F_\bullet; h_F, h_{F_\bullet}) = \sum_{i \geq 0} (-1)^i \mathrm{ch}_g(F_i, h_{F_i}) - \mathrm{ch}_g(F, h_F).$$

Applying the immersion formula of Bismut [6, Th. 0.1] to the immersion $\emptyset \hookrightarrow M_b$, $b \in U$, we get the following equation of complex-valued functions on U

(3.5)
$$\log \|\sigma\|^2_{Q, \lambda_G(F)}(g) = \sum_{i \geq 0} (-1)^i \log \|\sigma_i\|^2_{Q, \lambda_G(F_i)}(g)$$
$$+ \left[\pi_* \{ \mathrm{Td}_g(TM/B, h_{M/B}) \widetilde{\mathrm{ch}}_g(F, F_\bullet; h_F, h_{F_\bullet}) \} \right]^{(0)}$$
$$\equiv \sum_{i \geq 0} (-1)^i \log \|\sigma_i\|^2_{Q, \lambda_G(F_i)}(g) \mod C^\infty(U),$$

where we used [9, I, Th. 1.29 and Cor. 1.30] to identify the Bott–Chern current $T_g(F, F_\bullet; h_F, h_{F_\bullet})$ with the Bott–Chern class $\widetilde{\mathrm{ch}}_g(F, F_\bullet; h_F, h_{F_\bullet})$ in the first equality and $[\omega]^{(2d)}$ denotes the component of degree $2d$ of a differential form ω. Since the morphism $\pi \colon \pi^{-1}(U) \to U$ is smooth and since $h^0(M_b, F_i|_{M_b})$ is a constant function on U, we get $\log \|\sigma_i\|^2_{Q, \lambda_G(F_i)}(g) \in C^\infty(U)$ by [9, III, Th. 3.5]. This, together with (3.5), implies the result. $\qquad \square$

Remark 3.2. The curvature $-dd^c \log \|\sigma\|^2_{Q, \lambda_G(F)}(g)$ was computed by Bismut–Gillet–Soulé [9, Th. 0.1] when G is trivial and by Ma [21, Th. 2.12] when G is general and all direct image sheaves $R^q \pi_* F$, $q \geq 0$ are locally free. By Theorem 3.1, the curvature formula of Bismut-Gillet-Soulé-Ma

(3.6)
$$- dd^c \log \|\sigma\|^2_{Q, \lambda_G(F)}(g) = \left[\pi_* \{ \mathrm{Td}_g(TM/B, h_{M/B}) \, \mathrm{ch}_g(F, h_F) \} \right]^{(2)}$$

remains valid on B^o even if $R^q \pi_* F$ may not be locally free, because there is a dense Zariski open subset $U \subset B^o$ over which $h^q(F_b)$ is constant for all $q \geq 0$.

4. The singularity of equivariant Quillen metrics

Let $\lambda_G(\xi)$ be the equivariant determinant of the cohomologies of ξ. Then $\lambda_G(\xi)|_{C^o}$ is equipped with the equivariant Quillen metric $\| \cdot \|^2_{\lambda_G(\xi), Q}(\cdot)$ with respect to the G-invariant metrics $h_{\mathcal{X}/C}$, h_ξ. Let σ be an admissible holomorphic section of $\lambda_G(\xi)|_S$. By Theorem 3.1, $\log \|\sigma(s)\|^2_{\lambda_G(\xi), Q}(g) \in C^\infty(S^o)$ for $g \in G$. In this section, we determine the behavior of $\log \|\sigma(s)\|^2_{\lambda_G(\xi), Q}(g)$ as $s \to 0$.

4.1. **The non-twisted case.** Let $\Gamma \subset \mathcal{X} \times C$ be the graph of π. Then Γ is a smooth divisor on $\mathcal{X} \times C$ preserved by the G-action on $\mathcal{X} \times C$. Let $[\Gamma]$ be the G-equivariant holomorphic line bundle on $\mathcal{X} \times C$ associated to Γ. Let $\varsigma_\Gamma \in H^0(\mathcal{X} \times C, [\Gamma])$ be the canonical section of $[\Gamma]$ such that $\mathrm{div}(\varsigma_\Gamma) = \Gamma$. We identify $\mathcal{X}$ with Γ via the projection $\Gamma \to \mathcal{X}$.

Let $i \colon \Gamma \hookrightarrow \mathcal{X} \times C$ be the inclusion. Let $p_1 \colon \mathcal{X} \times C \to \mathcal{X}$ and $p_2 \colon \mathcal{X} \times C \to C$ be the projections. By the G-equivariances of i, p_1, p_2, we have the exact sequence of

G-equivariant coherent sheaves on $\mathcal{X} \times C$,

(4.1)
$$0 \longrightarrow \mathcal{O}_{\mathcal{X} \times C}([\Gamma]^{-1} \otimes p_1^* \xi) \xrightarrow{\otimes s_\Gamma} \mathcal{O}_{\mathcal{X} \times C}(p_1^* \xi) \longrightarrow i_* \mathcal{O}_\Gamma(p_1^* \xi) \longrightarrow 0.$$

Let $\lambda_G(p_1^* \xi)$, $\lambda_G([\Gamma]^{-1} \otimes p_1^* \xi)$, $\lambda_G(\xi)$ be the equivariant determinants of the direct images $R(p_2)_* \mathcal{O}_{\mathcal{X} \times C}(p_1^* \xi)$, $R(p_2)_* \mathcal{O}_{\mathcal{X} \times C}([\Gamma]^{-1} \otimes p_1^* \xi)$, $R\pi_* \mathcal{O}_\mathcal{X}(\xi)$, respectively. Under the isomorphism $p_1^* \xi|_\Gamma \cong \xi$ induced from the identification $p_1 \colon \Gamma \to \mathcal{X}$, the holomorphic vector bundle λ_G on C defined as

$$\begin{aligned}
\lambda_G &:= \lambda_G\left([\Gamma]^{-1} \otimes p_1^* \xi\right) \otimes \lambda_G(p_1^* \xi)^{-1} \otimes \lambda_G(\xi) = \prod_{W \in \widehat{G}} \lambda_W, \\
\lambda_W &:= \lambda_W\left([\Gamma]^{-1} \otimes p_1^* \xi\right) \otimes \lambda_W(p_1^* \xi)^{-1} \otimes \lambda_W(\xi)
\end{aligned}$$

carries the canonical holomorphic section $\sigma_{KM} = ((\sigma_{KM})_W)_{W \in \widehat{G}}$ such that $(\sigma_{KM})_W$ is identified with $1 \in H^0(C, \mathcal{O}_C)$ under the canonical isomorphism $\lambda_W \cong \mathcal{O}_C$; since λ_W is the determinant of the acyclic complex of coherent sheaves on C obtained as the W-component of the long exact sequence of direct images associated to (4.1), λ_W is canonically isomorphic to $\mathcal{O}_C$ ([6], [11], [18]). Then σ_{KM} is admissible.

Let $U \subset S$ be a relatively compact neighborhood of $0 \in \Delta$ and set $U^o := U \setminus \{0\}$. On $X = \pi^{-1}(S)$, we identify π (resp. $d\pi$) with $s \circ \pi$ (resp. $d(s \circ \pi)$). Hence $\pi \in \mathcal{O}(X)$ and $d\pi \in H^0(X, \Omega_X^1)$ in what follows.

Let $h_{[\Gamma]}$ be a G-invariant C^∞ Hermitian metric on $[\Gamma]$ such that

(4.2)
$$h_{[\Gamma]}(\varsigma_\Gamma, \varsigma_\Gamma)(w, t) = \begin{cases} |\pi(w) - t|^2 & \text{if} \quad (w, t) \in \pi^{-1}(U) \times U, \\ 1 & \text{if} \quad (w, t) \in (\mathcal{X} \setminus X) \times U \end{cases}$$

and let $h_{[\Gamma]^{-1}}$ be the metric on $[\Gamma]^{-1}$ induced from $h_{[\Gamma]}$.

For $g \in G$, let $\| \cdot \|_{Q, \lambda_G(\xi)}(g)$ be the equivariant Quillen metric on $\lambda_G(\xi)$ with respect to $h_{\mathcal{X}/C}$, h_ξ. Let $\| \cdot \|_{Q, \lambda_G([\Gamma]^{-1} \otimes p_1^* \xi)}$ (resp. $\| \cdot \|_{Q, \lambda_G(p_1^* \xi)}$) be the equivariant Quillen metric on $\lambda_G([\Gamma]^{-1} \otimes p_1^* \xi)$ (resp. $\lambda_G(p_1^* \xi)$) with respect to $h_\mathcal{X}$, $h_{[\Gamma]^{-1}} \otimes h_\xi$ (resp. $h_\mathcal{X}$, h_ξ). Let $\| \cdot \|_{Q, \lambda_G}$ be the equivariant Quillen metric on λ_G defined as the tensor product of those on $\lambda_G([\Gamma]^{-1} \otimes p_1^* \xi)$, $\lambda_G(p_1^* \xi)^{-1}$, $\lambda_G(\xi)$.

For the germ $(\pi \colon (\mathcal{X}, X_0) \to (S, 0), \xi)$, we define its topological invariant

$$\mathfrak{a}_g(X_0, \xi) = \int_{E_0 \cap \widetilde{\mathcal{X}}_H^g} \widetilde{\gamma}^* \{ \frac{1 - \mathrm{Td}(\mathcal{H})^{-1}}{c_1(\mathcal{H})} \} q^* \{ \mathrm{Td}_g(T\mathcal{X}) \mathrm{ch}_g(\xi) \} - \int_{\mathcal{X}_V^g \cap X_0} \mathrm{Td}_g(T\mathcal{X}) \mathrm{ch}_g(\xi).$$

Theorem 4.1. *For $g \in G$, the following identity of functions on S^o holds:*

$$\log \|\sigma_{KM}\|_{Q, \lambda_G}^2(g) \equiv_{\mathcal{B}} \mathfrak{a}_g(X_0, \xi) \log |s|^2.$$

Since $\mathcal{X}_H^1 = \mathcal{X}$ and $\mathcal{X}_V^1 = \emptyset$ for $g = 1$, we get [31, Th. 1.1] by Theorem 4.2.

Proof. We follow [7, Sect. 5], [30, Th. 6.3], [31, Th. 5.1]. The proof is quite parallel to that of [31, Th. 5.1]. The major differences come from the fact that $\mathcal{X}^g$ consists of the horizontal component $\mathcal{X}_H^g$ and vertical component $\mathcal{X}_V^g$ and these two components give different contributions to the singularity of $\log \|\sigma_{KM}\|_{Q, \lambda_G}^2(g)$.

Step 1 Let $[X_s] = [\Gamma]|_{X_s}$ be the holomorphic line bundle on $\mathcal{X}$ associated to the divisor X_s. The canonical section of $[X_s]$ is defined as $\varsigma_s = \varsigma_\Gamma|_{\mathcal{X} \times \{s\}} \in H^0(\mathcal{X}, [X_s])$. Then $\mathrm{div}(\varsigma_s) = X_s$. Let $i_s \colon X_s \hookrightarrow \mathcal{X}$ be the natural embedding. By (4.1), we get the exact sequence of G-equivariant coherent sheaves on $\mathcal{X}$,

(4.3)
$$0 \longrightarrow \mathcal{O}_\mathcal{X}([X_s]^{-1} \otimes \xi) \xrightarrow{\otimes \varsigma_s} \mathcal{O}_\mathcal{X}(\xi) \longrightarrow (i_s)_* \mathcal{O}_{X_s}(\xi) \longrightarrow 0,$$

which induces the canonical isomorphism $(\lambda_G)_s = \lambda_G([X_s]^{-1} \otimes \xi) \otimes \lambda_G(\xi)^{-1} \otimes \lambda_G(\xi_s)$.

Set $h_{[X_s]} = h_{[\Gamma]}|_{\mathcal{X} \times \{s\}}$, which is a G-invariant Hermitian metric on $[X_s]$. Let $h_{[X_s]^{-1}}$ be the G-invariant Hermitian metric on $[X_s]^{-1}$ induced from $h_{[X_s]}$.

Let $N_s = N_{X_s/\mathcal{X}}$ (resp. $N_s^* = N_{X_s/\mathcal{X}}^*$) be the normal (resp. conormal) bundle of X_s in $\mathcal{X}$. Then $d\pi|_{X_s} \in H^0(X_s, N_s^*)$ generates N_s^* for $s \in S^o$ and $d\pi|_{X_s}$ is G-invariant (cf. [6, Eq. (2.2)]). Let $a_{N_s^*}$ be the G-invariant Hermitian metric on N_s^* defined by $a_{N_s^*}(d\pi|_{X_s}, d\pi|_{X_s}) = 1$. Let a_{N_s} be the G-invariant Hermitian metric on N_s induced from $a_{N_s^*}$. We have the equality $c_1(N_s, a_{N_s}) = 0$ for $s \in U^o$. By [31, Proof of Th. 5.1 Step 1], the G-invariant metrics $h_{[X_s]^{-1}} \otimes h_\xi$ and h_ξ verify assumption (A) of Bismut [5, Def.1.5] with respect to a_{N_s} and $h_\xi|_{X_s}$.

Step 2 Let $\mathcal{E}_s$ be the exact sequence of G-equivariant holomorphic vector bundles on X_s defined as $\mathcal{E}_s \colon 0 \to TX_s \to T\mathcal{X}|_{X_s} \to N_s \to 0$. By [9, I], one has the Bott-Chern class $\widetilde{\mathrm{Td}}_g(\mathcal{E}_s; h_{X_s}, h_\mathcal{X}, a_{N_s}) \in \widetilde{A}_{X_s^g}$ such that

$$dd^c \widetilde{\mathrm{Td}}_g(\mathcal{E}_s; h_{X_s}, h_\mathcal{X}, a_{N_s}) = \mathrm{Td}_g(TX_s, h_{X_s})\, \mathrm{Td}(N_s, a_{N_s})|_{X_s^g} - \mathrm{Td}_g(T\mathcal{X}, h_\mathcal{X}).$$

Here we used the triviality of the G-action on $N_s|_{X_s^g}$ to get the equality $\mathrm{Td}_g(N_s, a_{N_s}) = \mathrm{Td}(N_s, a_{N_s})|_{X_s^g}$. Set $(X_H^g)_s := \mathcal{X}_H^g \cap X_s$. Applying the embedding formula of Bismut [6] (see also [7, Th. 5.6]) to the G-equivariant embedding $i_s \colon X_s \hookrightarrow \mathcal{X}$ and to the exact sequence (4.3), we get for all $s \in U^o$

(4.4)

$$\log \|\sigma_{KM}(s)\|_{Q,\lambda_G}^2(g) = \int_{(\mathcal{X}_V^g \times \{s\}) \amalg (\mathcal{X}_H^g \times \{s\})} -\frac{\mathrm{Td}_g(T\mathcal{X}, h_\mathcal{X})\, \mathrm{ch}_g(\xi, h_\xi)}{\mathrm{Td}([\Gamma], h_{[\Gamma]})} \log h_{[\Gamma]}(\varsigma_\Gamma, \varsigma_\Gamma)$$

$$- \int_{(X_H^g)_s} \frac{\widetilde{\mathrm{Td}}_g(\mathcal{E}_s; h_{X_s}, h_\mathcal{X}, a_{N_s})\, \mathrm{ch}_g(\xi, h_\xi)}{\mathrm{Td}(N_s, a_{N_s})} + C(g),$$

where $C(g)$ is a topological constant independent of $s \in U^o$. Here we used the triviality of the G-action on $[X_s]|_{X_s^g}$ and the explicit formula for the Bott-Chern current [10, Rem. 3.5, especially (3.23), Th. 3.15, Th. 3.17] to get the first term of the right hand side of (4.4). Substituting (4.2) and $c_1(N_s, a_{N_s}) = 0$ into (4.4), we get for $s \in U^o$

(4.5)

$$\log \|\sigma_{KM}(s)\|_{Q,\lambda_G}^2(g) \equiv_\mathcal{B} -\int_{\mathcal{X}_H^g \times \{s\}} \mathrm{Td}_g(T\mathcal{X}, h_\mathcal{X})\, \mathrm{ch}_g(\xi, h_\xi) \log |\pi - s|^2$$

$$- \int_{(\mathcal{X}_V^g \cap X_0) \times \{s\}} \mathrm{Td}_g(T\mathcal{X})\, \mathrm{ch}_g(\xi) \log |s|^2 - \int_{(X_H^g)_s} \widetilde{\mathrm{Td}}_g(\mathcal{E}_s; h_{X_s}, h_\mathcal{X}, a_{N_s})\, \mathrm{ch}_g(\xi, h_\xi)$$

$$\equiv_\mathcal{B} -\{\int_{\mathcal{X}_V^g \cap X_0} \mathrm{Td}_g(T\mathcal{X})\, \mathrm{ch}_g(\xi)\} \log |s|^2 - \int_{(X_H^g)_s} \widetilde{\mathrm{Td}}_g(\mathcal{E}_s; h_{X_s}, h_\mathcal{X}, a_{N_s})\, \mathrm{ch}_g(\xi, h_\xi),$$

where we used the equality $h_{[\Gamma]}(\varsigma_\Gamma, \varsigma_\Gamma)|_{X_0 \times \{s\}} = |s|^2$ and the fact $\mathcal{X}_V^g \cap \pi^{-1}(U) = \mathcal{X}_V^g \cap X_0$ to get the first equality and [31, Th. 9.1] to get the second equality.

Step 3 Let h_{N_s} be the Hermitian metric on N_s induced from $h_\mathcal{X}$ by the C^∞ isomorphism $N_s \cong (TX_s)^\perp$. Let $\widetilde{\mathrm{Td}}(N_s; a_{N_s}, h_{N_s}) \in \widetilde{A}_{X_s}$ be the Bott–Chern secondary class such that

$$dd^c \widetilde{\mathrm{Td}}(N_s; a_{N_s}, h_{N_s}) = \mathrm{Td}(N_s, a_{N_s}) - \mathrm{Td}(N_s, h_{N_s}).$$

Since G acts trivially on $N_s|_{X_s^g}$, we deduce from [15, I, Props. 1.3.2 and 1.3.4] that

(4.6)
$$\widetilde{\mathrm{Td}}_g(\mathcal{E}_s;\, h_{X_s}, h_{\mathcal{X}}, a_{N_s}) =$$
$$\widetilde{\mathrm{Td}}_g(\mathcal{E}_s;\, h_{X_s}, h_{\mathcal{X}}, h_{N_s}) + \mathrm{Td}_g(TX_s, h_{X_s})\,\widetilde{\mathrm{Td}}(N_s;\, a_{N_s}, h_{N_s}) =$$
$$\widetilde{\mathrm{Td}}_g(\mathcal{E}_s;\, h_{X_s}, h_{\mathcal{X}}, h_{N_s}) + \gamma^*\mathrm{Td}_g(\mathcal{U}, h_{\mathcal{U}})\,\nu^*\{\frac{1 - \mathrm{Td}(-c_1(\mathcal{L}, h_{\mathcal{L}}))}{-c_1(\mathcal{L}, h_{\mathcal{L}})}\}\, \log\|d\pi\|^2\Big|_{(X_H^g)_s}.$$

Here we used [31, Eq. (13)] and the relation $(TX_s, h_{X_s}) = \gamma^*(\mathcal{U}, h_{\mathcal{U}})|_{X_s}$ to get the second equality. Since
$$(\mathcal{E}_s, h_{X_s}, h_{\mathcal{X}}, h_{N_s}) = \gamma^*(\mathcal{S}^\vee, h_{\mathcal{U}}, (\Pi^\vee)^*h_{\mathcal{X}}, h_{\mathcal{H}})|_{X_s},$$
we get by the functorial property of the Bott–Chern class [9, I]

(4.7) $\qquad \widetilde{\mathrm{Td}}_g(\mathcal{E}_s;\, h_{X_s}, h_{\mathcal{X}}, h_{N_s}) = \gamma^*\widetilde{\mathrm{Td}}_g(\mathcal{S}^\vee;\, h_{\mathcal{U}}, (\Pi^\vee)^*h_{\mathcal{X}}, h_{\mathcal{H}})|_{(X_H^g)_s}.$

Substituting (4.7) into (4.6), we get

(4.8)
$$\widetilde{\mathrm{Td}}_g(\mathcal{E}_s;\, h_{X_s}, h_{\mathcal{X}}, a_{N_s}) = \gamma^*\widetilde{\mathrm{Td}}_g(\mathcal{S}^\vee;\, h_{\mathcal{U}}, (\Pi^\vee)^*h_{\mathcal{X}}, h_{\mathcal{H}})|_{(X_H^g)_s} +$$
$$\gamma^*\mathrm{Td}_g(\mathcal{U}, h_{\mathcal{U}})\,\nu^*\{\frac{1 - \mathrm{Td}(-c_1(\mathcal{L}, h_{\mathcal{L}}))}{-c_1(\mathcal{L}, h_{\mathcal{L}})}\}\, \log\|d\pi\|^2|_{(X_H^g)_s}.$$

Substituting (4.8) into (4.5), we get by the same argument as in [31, p.74 l.1-l.13]

(4.9)
$$\log\|\sigma_{KM}\|^2_{Q,\lambda_G}(g) \equiv_{\mathcal{B}} -\{\int_{\mathcal{X}_V^g \cap X_0} \mathrm{Td}_g(T\mathcal{X})\,\mathrm{ch}_g(\xi)\}\, \log|s|^2$$
$$- (\pi|_{\mathcal{X}_H^g})_* \left[\gamma^*\widetilde{\mathrm{Td}}_g(\mathcal{S}^\vee;\, h_{\mathcal{U}}, (\Pi^\vee)^*h_{\mathcal{X}}, h_{\mathcal{H}})\,\mathrm{ch}_g(\xi, h_\xi)\right]^{(0)}$$
$$- (\pi|_{\mathcal{X}_H^g})_* \left[\gamma^*\mathrm{Td}_g(\mathcal{U}, h_{\mathcal{U}})\,\nu^*\{\frac{1 - \mathrm{Td}(-c_1(\mathcal{L}, h_{\mathcal{L}}))}{-c_1(\mathcal{L}, h_{\mathcal{L}})}\}\,\mathrm{ch}_g(\xi, h_\xi)\, \log\|d\pi\|^2\right]^{(0)}$$
$$\equiv_{\mathcal{B}} -\{\int_{\mathcal{X}_V^g \cap X_0} \mathrm{Td}_g(T\mathcal{X})\,\mathrm{ch}_g(\xi)\}\, \log|s|^2$$
$$+ (\widetilde{\pi}|_{\mathcal{X}_H^g})_* \left[\widetilde{\gamma}^*\mathrm{Td}_g(\mathcal{U}, h_{\mathcal{U}})\,\widetilde{\nu}^*\{\frac{\mathrm{Td}(-c_1(\mathcal{L}, h_{\mathcal{L}})) - 1}{-c_1(\mathcal{L}, h_{\mathcal{L}})}\}\, q^*\mathrm{ch}_g(\xi, h_\xi)\, (q^*\log\|d\pi\|^2)\right]^{(0)}$$

where we set $\widetilde{\pi} := \pi \circ q$. By [31, Cor. 4.6] applied to the last line of (4.9), we get

$$\log\|\sigma_{KM}\|^2_{Q,\lambda_G}(g) \equiv_{\mathcal{B}}$$
$$[\int_{\widetilde{\mathcal{X}}_H^g \cap E_0} \widetilde{\gamma}^*\{\mathrm{Td}_g(\mathcal{U})\frac{\mathrm{Td}(\mathcal{H}) - 1}{c_1(\mathcal{H})}\}q^*\mathrm{ch}_g(\xi) - \int_{\mathcal{X}_V^g \cap X_0} \mathrm{Td}_g(T\mathcal{X})\,\mathrm{ch}_g(\xi)]\, \log|s|^2$$
$$= [\int_{\widetilde{\mathcal{X}}_H^g \cap E_0} \widetilde{\gamma}^*\{\frac{1 - \mathrm{Td}(\mathcal{H})^{-1}}{c_1(\mathcal{H})}\}q^*\{\mathrm{Td}_g(T\mathcal{X})\mathrm{ch}_g(\xi)\}]\, \log|s|^2$$
$$- \{\int_{\mathcal{X}_V^g \cap X_0} \mathrm{Td}_g(T\mathcal{X})\,\mathrm{ch}_g(\xi)\}\, \log|s|^2 = \mathfrak{a}_g(X_0, \xi)\, \log|s|^2.$$

Here the first equality follows from the identity $\mathrm{Td}_g(\mathcal{U})\mathrm{Td}(\mathcal{H}) = (\Pi^\vee)^*\mathrm{Td}_g(T\mathcal{X})$, which is deduced from the exact sequence $0 \to \mathcal{U} \to (\Pi^\vee)^*T\mathcal{X} \to \mathcal{H} \to 0$ on $\mathbf{P}(T\mathcal{X})^\vee$. This completes the proof. $\qquad\square$

Theorem 4.2. *For $g \in G$, the following identity holds:*

$$\log \|\sigma\|^2_{Q,\lambda_G(\xi)}(g) \equiv_{\mathcal{B}} \mathfrak{a}_g(X_0, \xi) \log |s|^2.$$

Proof. There exist admissible holomorphic sections

$$\alpha = (\alpha_W)_{W \in \widehat{G}} \in \Gamma(U, \lambda_G(p_1^* \xi)), \qquad \beta = (\beta_W)_{W \in \widehat{G}} \in \Gamma(U, \lambda_G([\Gamma]^{-1} \otimes p_1^* \xi))$$

such that $\sigma_{KM} = \beta \otimes \alpha^{-1} \otimes \sigma$ on S, i.e., $(\sigma_{KM})_W = \beta_W \otimes \alpha_W^{-1} \otimes \sigma_W$ for all $W \in \widehat{G}$. Then

$$\log \|\sigma\|^2_{Q,\lambda_G(\xi)}(g) = \log \|\sigma_{KM}\|^2_{Q,\lambda_G}(g) + \log \|\alpha\|^2_{Q,\lambda_G(p_1^*\xi)}(g) - \log \|\beta\|^2_{Q,\lambda_G([\Gamma]\otimes p_1^*\xi)}(g)$$
$$\equiv_{\mathcal{B}} \mathfrak{a}_g(X_0, \xi) \log |s|^2$$

by Theorems 3.1 and 4.1. This proves the theorem. $\qquad \square$

Corollary 4.3. *The following equation of $(1,1)$-currents on S holds:*

$$-dd^c \log \|\sigma\|^2_{Q,\lambda_G(\xi)}(g) = \pi_* \left[\mathrm{Td}_g(T\mathcal{X}/C, h_{\mathcal{X}/C}) \, \mathrm{ch}_g(\xi, h_\xi) \right]^{(1,1)} - \mathfrak{a}_g(X_0, \xi) \, \delta_0.$$

Proof. The result follows from the curvature formula (3.6) and Theorem 4.2. $\quad \square$

4.2. **The case of adjoint bundles twisted by the relative canonical bundle.**
We set $\Omega^1_{\mathcal{X}/C} := \Omega^1_{\mathcal{X}}/\pi^* \Omega^1_C$ and $\Omega^q_{\mathcal{X}/C} := \bigwedge^q \Omega^1_{\mathcal{X}/C}$. Let $\omega_{\mathcal{X}} := \Omega^{n+1}_{\mathcal{X}}$ be the canonical line bundle of $\mathcal{X}$ and let $\omega_{\mathcal{X}/C} := \Omega^{n+1}_{\mathcal{X}} \otimes (\pi^* \Omega^1_C)^{-1}$ be the relative canonical line bundle of $\pi \colon \mathcal{X} \to C$, which are identified with the dualizing sheaf of $\mathcal{X}$ and the relative dualizing sheaf of $\pi \colon \mathcal{X} \to C$ respectively. On $\mathcal{X} \setminus \Sigma_\pi$, there is a canonical isomorphism $\Omega^n_{\mathcal{X}/C} \cong \omega_{\mathcal{X}/C}$ induced by the short exact sequence $0 \to \pi^* \Omega^1_C \to \Omega^1_{\mathcal{X}} \to \Omega^1_{\mathcal{X}/C} \to 0$. The holomorphic vector bundle $\Omega^q_{\mathcal{X}/C}$ on $\mathcal{X} \setminus \Sigma_\pi$ is equipped with the Hermitian metric induced from $h_{\mathcal{X}/C}$. Since $\pi^* \Omega^1_C \subset \Omega^1_{\mathcal{X}}$, $\omega_{\mathcal{X}}$ and $\omega_{\mathcal{X}/C}$ are equipped with the Hermitian metrics $h_{\omega_{\mathcal{X}}}$ and $h_{\omega_{\mathcal{X}/C}}$ induced from $h_{\mathcal{X}}$, respectively. Then the canonical isomorphism $\Omega^n_{\mathcal{X}/C} \cong \omega_{\mathcal{X}/C}$ is an isometry.

For $g \in G$, let $\|\cdot\|_{Q,\lambda_G(\omega_{\mathcal{X}/C}(\xi))}(g)$ be the equivariant Quillen metric on $\lambda_G(\omega_{\mathcal{X}/C}(\xi))$ with respect to $h_{\mathcal{X}/C}$, h_ξ, $h_{\omega_{\mathcal{X}/C}}$. Let ς be a nowhere vanishing holomorphic section of $\lambda(\omega_{\mathcal{X}/C}(\xi))$ on S.

Theorem 4.4. *For $g \in G$, the following identity of functions on S^o holds:*

$$\log \|\varsigma\|^2_{Q,\lambda_G(\omega_{\mathcal{X}/C}(\xi))}(g) \equiv_{\mathcal{B}} \alpha_g(X_0, \omega_{X/S}(\xi)) \log |s|^2.$$

Proof. By Theorem 4.2 applied to $\omega_{\mathcal{X}}(\xi)$, we get on S

$$(4.10) \qquad \log \|\varsigma\|^2_{Q,\lambda_G(\omega_{\mathcal{X}}(\xi))}(g) \equiv_{\mathcal{B}} \mathfrak{a}_g(X_0, \omega_{\mathcal{X}}(\xi)) \log |s|^2.$$

Since $\pi^* \Omega^1_C$ is generated by $d\pi$ on $X \setminus \Sigma_\pi$, we get an identification $(\omega_{\mathcal{X}/C}, h_{\omega_{\mathcal{X}/C}}) = (\omega_{\mathcal{X}}, \|d\pi\|^{-2} h_{\omega_{\mathcal{X}}})$ on X. We set $h_{\omega_{\mathcal{X}}(\xi)} := h_{\omega_{\mathcal{X}}} \otimes h_\xi$ and $h_{\omega_{\mathcal{X}/C}(\xi)} := h_{\omega_{\mathcal{X}/C}} \otimes h_\xi$. By the anomaly formula [6, Th. 2.5], [9, I, Th. 0.3], we get

(4.11)

$$\log \|\varsigma\|^2_{Q,\lambda_G(\omega_{\mathcal{X}/C}(\xi))}(g) = \log \|\varsigma\|^2_{Q,\lambda_G(\omega_{\mathcal{X}}(\xi))}(g) + \log \frac{\|\cdot\|^2_{Q,\lambda_G(\omega_{\mathcal{X}/C}(\xi))}(g)}{\|\cdot\|^2_{Q,\lambda_G(\omega_{\mathcal{X}}(\xi))}(g)}$$

$$= \log \|\varsigma\|^2_{Q,\lambda_G(\omega_{\mathcal{X}}(\xi))}(g) + \pi_*(\mathrm{Td}_g(T\mathcal{X}/C, h_{\mathcal{X}/C}) \widetilde{\mathrm{ch}}_g(\omega_{\mathcal{X}}(\xi); h_{\omega_{\mathcal{X}}(\xi)}, h_{\omega_{\mathcal{X}/C}(\xi)}))^{(0)}.$$

Here $\widetilde{\mathrm{ch}}_g(\omega_{\mathcal{X}}(\xi); h_{\omega_{\mathcal{X}}(\xi)}, h_{\omega_{\mathcal{X}/C}(\xi)})$ is the Bott-Chern class such that

$$dd^c \widetilde{\mathrm{ch}}_g(\omega_{\mathcal{X}}(\xi); h_{\omega_{\mathcal{X}}(\xi)}, h_{\omega_{\mathcal{X}/C}(\xi)}) = \mathrm{ch}_g(\omega_{\mathcal{X}}(\xi), h_{\omega_{\mathcal{X}}(\xi)}) - \mathrm{ch}_g(\omega_{\mathcal{X}}(\xi), h_{\omega_{\mathcal{X}/C}(\xi)})$$

Since $(\omega_{\mathcal{X}/C}(\xi), h_{\omega_{\mathcal{X}/C}(\xi)}) = (\omega_{\mathcal{X}}(\xi), \|d\pi\|^{-2} h_{\omega_{\mathcal{X}}(\xi)})$ and $-dd^c \log \|d\pi\|^2 = \gamma^* c_1(\mathcal{L}, h_{\mathcal{L}})$, we get by [15, I, (1.2.5.1), (1.3.1.2) (see also [13, Eqs. (3.7), (5.5)])

(4.12)

$$\widetilde{\mathrm{ch}}_g(\omega_{\mathcal{X}}(\xi); h_{\omega_{\mathcal{X}}(\xi)}, h_{\omega_{\mathcal{X}/C}(\xi)}) = \widetilde{\mathrm{ch}}_g(\omega_{\mathcal{X}}; h_{\omega_{\mathcal{X}}}, \|d\pi\|^{-2} h_{\omega_{\mathcal{X}}}) \mathrm{ch}_g(\xi, h_\xi)$$

$$= \mathrm{ch}_g(\omega_{\mathcal{X}}, h_{\omega_{\mathcal{X}}}) \frac{e^{-\gamma^* c_1(\mathcal{L}, h_{\mathcal{L}})} - 1}{-\gamma^* c_1(\mathcal{L}, h_{\mathcal{L}})} (-\log \|d\pi\|^2) \wedge \mathrm{ch}_g(\xi, h_\xi)$$

$$= -\mathrm{ch}_g(\omega_{\mathcal{X}}(\xi), h_{\omega_{\mathcal{X}}(\xi)}) \frac{1 - e^{-\gamma^* c_1(\mathcal{L}, h_{\mathcal{L}})}}{\gamma^* c_1(\mathcal{L}, h_{\mathcal{L}})} \log \|d\pi\|^2.$$

Since $\mathrm{Td}_g(T\mathcal{X}/C, h_{\mathcal{X}/C}) = \gamma^* \mathrm{Td}_g(\mathcal{U})$, we get by (4.11) and [31, Cor. 4.6]

(4.13)

$$\log \frac{\|\cdot\|^2_{Q, \lambda_G(\omega_{\mathcal{X}/C}(\xi))}(g)}{\|\cdot\|^2_{Q, \lambda_G(\omega_{\mathcal{X}}(\xi))}(g)}$$

$$= -\pi_* \{ \mathrm{Td}_g(T\mathcal{X}/C, h_{\mathcal{X}/C}) \mathrm{ch}_g(\omega_{\mathcal{X}}(\xi), h_{\omega_{\mathcal{X}}} \otimes h_\xi) \frac{1 - e^{-\gamma^* c_1(\mathcal{L}, h_{\mathcal{L}})}}{\gamma^* c_1(\mathcal{L}, h_{\mathcal{L}})} \log \|d\pi\|^2 \}^{(0)}$$

$$\equiv_{\mathcal{B}} -\{ \int_{E_0 \cap \widetilde{\mathcal{X}}^g_H} \widetilde{\gamma}^* \mathrm{Td}_g(\mathcal{U}) q^* \mathrm{ch}_g(\omega_{\mathcal{X}}(\xi)) \frac{1 - e^{-\widetilde{\gamma}^* c_1(\mathcal{L})}}{\widetilde{\gamma}^* c_1(\mathcal{L})} \} \log |s|^2$$

$$\equiv_{\mathcal{B}} -[\int_{E_0 \cap \widetilde{\mathcal{X}}^g_H} \widetilde{\gamma}^* \{ \mathrm{Td}_g(\mathcal{U}) \frac{e^{c_1(\mathcal{H})} - 1}{c_1(\mathcal{H})} \} q^* \mathrm{ch}_g(\omega_{\mathcal{X}}(\xi))] \log |s|^2.$$

By (4.10) and (4.13), we get

(4.14)

$$\log \|\varsigma\|^2_{Q, \lambda_G(\omega_{\mathcal{X}/C}(\xi))}(g) \equiv_{\mathcal{B}}$$

$$[\mathfrak{a}_g(\pi, X_0, \omega_{\mathcal{X}}(\xi)) - \int_{E_0 \cap \widetilde{\mathcal{X}}^g_H} \widetilde{\gamma}^* \{ \mathrm{Td}_g(\mathcal{U}) \frac{e^{c_1(\mathcal{H})} - 1}{c_1(\mathcal{H})} \} q^* \mathrm{ch}_g(\omega_{\mathcal{X}}(\xi))] \log |s|^2$$

$$= [\int_{E_0 \cap \widetilde{\mathcal{X}}^g_H} \widetilde{\gamma}^* \{ \mathrm{Td}_g(\mathcal{U}) \left(\frac{\mathrm{Td}(\mathcal{H}) - 1}{c_1(\mathcal{H})} - \frac{e^{c_1(\mathcal{H})} - 1}{c_1(\mathcal{H})} \right) \} q^* \mathrm{ch}_g(\omega_{\mathcal{X}}(\xi))] \log |s|^2$$

$$- \{ \int_{\mathcal{X}^g_V \cap X_0} \mathrm{Td}_g(T\mathcal{X}) \mathrm{ch}_g(\omega_{\mathcal{X}}(\xi)) \} \log |s|^2$$

$$= [\int_{E_0 \cap \widetilde{\mathcal{X}}^g_H} \widetilde{\gamma}^* \{ \mathrm{Td}_g(\mathcal{U}) \mathrm{Td}(\mathcal{H}) \left(\frac{1 - \mathrm{Td}(\mathcal{H})^{-1}}{c_1(\mathcal{H})} - \frac{e^{c_1(\mathcal{H})}}{\mathrm{Td}(\mathcal{H})^2} \right) \} q^* \mathrm{ch}_g(\omega_{\mathcal{X}}(\xi))] \log |s|^2$$

$$- \{ \int_{\mathcal{X}^g_V \cap X_0} \mathrm{Td}_g(T\mathcal{X}) \mathrm{ch}_g(\omega_{\mathcal{X}}(\xi)) \} \log |s|^2$$

$$= [\int_{E_0 \cap \widetilde{\mathcal{X}}^g_H} \widetilde{\gamma}^* \left(\frac{\mathrm{Td}(\mathcal{H}^\vee)^{-1} - 1}{c_1(\mathcal{H}^\vee)} \right) q^* \{ \mathrm{Td}_g(T\mathcal{X}) \mathrm{ch}_g(\omega_{\mathcal{X}}(\xi)) \} \}] \log |s|^2$$

$$- \{ \int_{\mathcal{X}^g_V \cap X_0} \mathrm{Td}_g(T\mathcal{X}) \mathrm{ch}_g(\omega_{\mathcal{X}}(\xi))] \log |s|^2 = \alpha_g(X_0, \omega_{\mathcal{X}/C}(\xi)) \log |s|^2.$$

Here the fourth equality follows from the identities $\mathrm{Td}_g(\mathcal{U}) \mathrm{Td}(\mathcal{H}) = (\Pi^\vee)^* \mathrm{Td}_g(T\mathcal{X})$ and

$$\frac{1 - \mathrm{Td}(x)^{-1}}{x} - e^x \mathrm{Td}(x)^{-2} = \frac{\mathrm{Td}(-x)^{-1} - 1}{-x}, \qquad \mathrm{Td}(x) = x/(1 - e^{-x}).$$

This completes the proof.   □

4.3. **Compatibility with the Serre duality.** We check the compatibility of Theorems 4.2 and 4.4 with the Serre duality. There exists an exact sequence of G-equivariant holomorphic vector bundles $0 \to \xi \to \xi_0 \to \cdots \to \xi_m \to 0$ on $\mathcal{X}$ such that $H^q(X_s, \xi_i|_{X_s}) = 0$ for all $i \geq 0$, $q > 0$, $s \in C$. We get the corresponding long exact sequence of G-equivariant holomorphic vector bundles $0 \to \omega_{\mathcal{X}/C}(\xi_m^\vee) \to \cdots \to \omega_{\mathcal{X}/C}(\xi_0^\vee) \to \omega_{\mathcal{X}/C}(\xi^\vee) \to 0$ on $\mathcal{X}$. Since $R^q\pi_*\omega_{\mathcal{X}/C}(\xi_i^\vee) = 0$ for $q \neq n$ by the fiberwise Serre duality, we get for all $q \geq 0$

$$R^q\pi_*\omega_{\mathcal{X}/C}(\xi^\vee) = \frac{\ker\{R^n\pi_*\omega_{\mathcal{X}/C}(\xi_{n-q}^\vee) \to R^n\pi_*\omega_{\mathcal{X}/C}(\xi_{n-q-1}^\vee)\}}{\mathrm{Im}\{R^n\pi_*\omega_{\mathcal{X}/C}(\xi_{n-q+1}^\vee) \to R^n\pi_*\omega_{\mathcal{X}/C}(\xi_{n-q}^\vee)\}}.$$

Hence $\lambda_W(\omega_{\mathcal{X}/C}(\xi^\vee))^{(-1)^n} \cong \bigotimes_{i \geq 0} \lambda_W(R^n\pi_*\omega_{\mathcal{X}/C}(\xi_i^\vee))^{(-1)^i}$ for $W \in \widehat{G}$. Since $\lambda_W(\xi)^\vee \cong \bigotimes_{i \geq 0}(\lambda_W(\pi_*\xi_i)^\vee)^{(-1)^i} \cong \bigotimes_{i \geq 0} \lambda_W(R^n\pi_*\omega_{\mathcal{X}/C}(\xi_i^\vee))^{(-1)^i}$, there is a canonical isomorphism $\lambda_W(\xi) \otimes \lambda_W(\omega_{\mathcal{X}/C}(\xi^\vee))^{(-1)^n} \cong \mathcal{O}_C$. Let $\mathbf{1}_W$ be the canonical section of $\lambda_W(\xi) \otimes \lambda_W(\omega_{\mathcal{X}/C}(\xi^\vee))^{(-1)^n}$ corresponding to $1 \in H^0(C, \mathcal{O}_C)$. Then $\mathbf{1} = (\mathbf{1}_W)_{W \in \widehat{G}}$ is the canonical section of $\lambda_G(\xi) \otimes \lambda_G(\omega_{\mathcal{X}/C}(\xi^\vee))^{(-1)^n}$. By the same argument as in [16, p.27], we get $\log\|\mathbf{1}\|_{Q,\lambda_G(\xi)\otimes\lambda_G(\omega_{\mathcal{X}/C}(\xi^\vee))^{(-1)^n}}(g) = 0$ as a function on C^o, which, together with Theorems 4.2 and 4.4, implies the equality

$$(4.15) \qquad \mathfrak{a}_g(X_0, \xi) + (-1)^n\alpha_g(X_0, \omega_{X/S}(\xi^\vee)) = 0.$$

We check (4.15). Let $T\mathcal{X}|_{\mathcal{X}^g} = \oplus_j E(\theta_j)$ be the decomposition into the eigenbundles with respect to the g-action. Let $\lambda_0, \ldots, \lambda_n$ be the Chern roots of $T\mathcal{X}|_{\mathcal{X}^g}$. We may assume that the element g acts as the multiplication by e^{θ_i} on the line bundle corresponding to λ_i. Since $e^{-\lambda_i}[\lambda_i/(1 - e^{-\lambda_i})] = (-\lambda_i/1 - e^{-(-\lambda_i)})$ and $e^{-\lambda_i+\theta_i}[1/(1 - e^{-\lambda_i+\theta_i})] = -(1/1 - e^{-(-\lambda_i+\theta_i)})$, we get

$$\mathrm{Td}_g(T\mathcal{X})\mathrm{ch}_g(\omega_\mathcal{X} \otimes \xi^\vee)|_{\mathcal{X}_i^g} = (-1)^{n+1-\dim \mathcal{X}_i^g}\mathrm{Td}_g(T\mathcal{X}^\vee)\mathrm{ch}_g(\xi^\vee)|_{\mathcal{X}_i^g}$$

for every connected component $\mathcal{X}_i^g$ of $\mathcal{X}^g$. Set $d_i := \dim \mathcal{X}_i^g$. Then

(4.16)
$$(-1)^n\alpha_g(X_0, \omega_{\mathcal{X}/C}(\xi^\vee))$$
$$= (-1)^n \sum_{i \in I} \int_{E_0 \cap (\widetilde{\mathcal{X}}_H^g)_i} \widetilde{\gamma}^*\{\frac{\mathrm{Td}(\mathcal{H}^\vee)^{-1} - 1}{c_1(\mathcal{H}^\vee)}\} q^*\{\mathrm{Td}_g(T\mathcal{X})\mathrm{ch}_g(\omega_\mathcal{X} \otimes \xi^\vee)\}$$
$$\quad - (-1)^n \sum_{j \in J} \int_{(\mathcal{X}_V^g)_j \cap X_0} \mathrm{Td}_g(T\mathcal{X})\mathrm{ch}_g(\omega_\mathcal{X} \otimes \xi^\vee)$$
$$= \sum_{i \in I} (-1)^{d_i+1} \int_{E_0 \cap (\widetilde{\mathcal{X}}_H^g)_i} \widetilde{\gamma}^*\{\frac{\mathrm{Td}(\mathcal{H}^\vee)^{-1} - 1}{c_1(\mathcal{H}^\vee)}\} q^*\{\mathrm{Td}_g(T\mathcal{X}^\vee)\mathrm{ch}_g(\xi^\vee)\}$$
$$\quad - \sum_{j \in J} (-1)^{d_j+1} \int_{(\mathcal{X}_V^g)_j \cap X_0} \mathrm{Td}_g(T\mathcal{X}^\vee)\mathrm{ch}_g(\xi^\vee)$$
$$= \sum_{i \in I} (-1)^{d_i+1} \int_{E_0 \cap (\widetilde{\mathcal{X}}_H^g)_i} \sum_{e \geq 0}(-1)^e \left[\widetilde{\gamma}^*\{\frac{\mathrm{Td}(\mathcal{H})^{-1} - 1}{c_1(\mathcal{H})}\} q^*\{\mathrm{Td}_g(T\mathcal{X})\mathrm{ch}_g(\xi)\}\right]^{(2e)}$$
$$\quad - \sum_{j \in J} (-1)^{d_j+1} \int_{(\mathcal{X}_V^g)_j \cap X_0} \sum_{e \geq 0}(-1)^e[\mathrm{Td}_g(T\mathcal{X})\mathrm{ch}_g(\xi)]^{(2e)},$$

where $\mathcal{X}_H^g = \mathrm{II}_{i \in I}(\mathcal{X}_H^g)_i$ and $\mathcal{X}_V^g = \mathrm{II}_{j \in J}(\mathcal{X}_V^g)_j$ are the decompositions into the connected components. Since $\mathcal{X}_H^g$ intersects X_0 properly, we get $\dim E_0 \cap (\widetilde{\mathcal{X}}_H^g)_i = \dim(\widetilde{\mathcal{X}}_H^g)_i - 1$. On the other hand, since $\mathcal{X}_V^g$ is contained in the singular fiber of π, we get $\dim(\mathcal{X}_V^g)_j \cap X_0 = \dim(\mathcal{X}_V^g)_j$ when $(\mathcal{X}_V^g)_j \cap X_0 \neq \emptyset$. By (4.16), we get (4.15)

$$(-1)^n \alpha_g(X_0, \omega_{\mathcal{X}/C}(\xi^\vee)) = \sum_{i \in I} \int_{E_0 \cap (\widetilde{\mathcal{X}}_H^g)_i} \widetilde{\gamma}^* \{ \frac{\mathrm{Td}(\mathcal{H})^{-1} - 1}{c_1(\mathcal{H})} \} q^* \{ \mathrm{Td}_g(T\mathcal{X}) \mathrm{ch}_g(\xi) \}$$

$$+ \sum_{j \in J} \int_{(\mathcal{X}_V^g)_j \cap X_0} \mathrm{Td}_g(T\mathcal{X}) \mathrm{ch}_g(\xi) = -\mathfrak{a}_g(X_0, \xi).$$

5. Nakano semi-positive vector bundles

5.1. **Semi-positivity and semi-negativity of vector bundles.** Let M be a connected complex manifold of dimension $n + 1$ and let $E \to M$ be a holomorphic vector bundle of rank r equipped with a Hermitian metric h_E. Let $R^E = (\nabla^E)^2$ be the curvature of E, where ∇^E is the holomorphic Hermitian connection of (E, h_E). Write $h_E(i R^E(\cdot), \cdot) = \sum_{a,b,\alpha,\beta} R_{\alpha\bar{\beta}a\bar{b}} (e_\alpha^\vee \otimes \bar{e}_\beta^\vee) \otimes (\theta_a \wedge \bar{\theta}_b)$, where $\{e_\alpha^\vee\}$ (resp. $\{\theta_a\}$) is a local *unitary* frame of $E^\vee$ (resp. Ω_M^1). Then (E, h_E) is said to be *Nakano semi-positive* if $\sum_{a,b,\alpha,\beta} R_{a\bar{b}\alpha\bar{\beta}} \zeta_a^\alpha \bar{\zeta}_b^\beta \geq 0$ for all $(\zeta_a^\alpha) \in \mathbf{C}^{r(n+1)}$. Similarly, (E, h_E) is said to be *semi-negative in the dual Nakano sense* if $\sum_{a,b,\alpha,\beta} R_{a\bar{b}\alpha\bar{\beta}} \zeta_b^\alpha \bar{\zeta}_a^\beta \leq 0$ for all $(\zeta_b^\alpha) \in \mathbf{C}^{r(n+1)}$. Note the difference of indices in these two definitions. By [26, Lemma 4.3], (E, h_E) is Nakano semi-positive if and only if $(E^\vee, h_{E^\vee})$ is semi-negative in the dual Nakano sense, where $h_{E^\vee}$ is the metric on $E^\vee$ induced from h_E. We thank Professor Shigeharu Takayama for pointing out the fact that the dual of a Nakano semi-positive vector bundle is *not necessarily* Nakano semi-negative but semi-negative in the dual Nakano sense.

5.2. **Some results of Takegoshi for Nakano semi-positive vector bundles.** We recall two results for Nakano semi-positive vector bundles from [28]. Let $\Delta \subset \mathbf{C}$ be the unit disc. Let $\pi \colon M \to \Delta$ be a proper surjective holomorphic map with critical locus Σ and set $M_t := \pi^{-1}(t)$ for $t \in \Delta$. Assume that M is a Kähler manifold with Kähler form κ_M. Set $\kappa_{M_t} := \kappa_M|_{M_t}$ and $h_{E_t} := h_E|_{M_t}$.

Theorem 5.1. *Assume that (E, h_E) is Nakano semi-positive. For every $u \in H^q(M, \Omega_M^{n+1}(E))$, there exists $\sigma \in H^0(M, \Omega_M^{n+1-q}(E))$ with*

$$u = [\sigma \wedge \kappa_M^q], \qquad (\pi^* dt) \wedge \sigma = 0.$$

In particular, there exists $v \in H^0(M \setminus \Sigma, \Omega_{M/\Delta}^{n-q}(E))$ with

$$u|_{M \setminus \Sigma} = [v \wedge \kappa_M^q \wedge (\pi^* dt)].$$

For $t \in \Delta \setminus \pi(\Sigma)$, $v|_{M_t} \wedge \kappa_{M_t}^q \in A_{M_t}^{n,q}(\Omega_{M_t}^n(E|_{M_t}))$ is a harmonic form with respect to κ_{M_t}, h_{E_t}.

Proof. See [28, Th. 5.2 (i), (ii)]. Notice that $v|_{M_t} \wedge \kappa_{M_t}^q = *(v|_{M_t})$ is a harmonic form with respect to κ_{M_t}, h_{E_t}, since $v|_{M_t} \in H^0(M_t, \Omega_{M_t}^{n-q})$ is holomorphic and since the Hodge star operator preserves harmonic forms. $\square$

Theorem 5.1 and its extension by Mourougane-Takayama [23, Prop. 4.4] shall play a key role in Sects.6 and 9 to prove Theorem 1.1.

Lemma 5.2. *If (E, h_E) is Nakano semi-positive, then $R^q\pi_*\Omega_M^{n+1}(E)$ is locally free for all $q \geq 0$.*

Proof. Since $R^q\pi_*\Omega_M^{n+1}(E)$ is torsion free by [28, Th. 6.5 (i)] and since $\dim S = 1$, we get the result. $\square$

We refer to [26, Sect. 4], [12, Chap. VII] for more about various notions of positivity and negativity of vector bundles and [3], [22], [23], [28] for more about the direct images of Nakano semi-positive vector bundles twisted by the relative canonical bundle.

6. Asymptotic behavior of equivariant analytic torsion

6.1. **Set up.** Let $\kappa_{\mathcal{X}}$ be the Kähler form of $h_{\mathcal{X}}$. In the rest of this paper, we assume that (ξ, h_ξ) *is Nakano semi-positive on* X *and that* $(S, 0) \cong (\Delta, 0)$. By Lemma 5.2, $R^q\pi_*\omega_{X/S}(\xi)$ is locally free on S. By shrinking S if necessary, we may also assume that $R^q\pi_*\omega_{X/S}(\xi)$ is a free $\mathcal{O}_S$-module on S. By the G-equivariance of $R^q\pi_*\omega_{X/S}(\xi)$, $\mathrm{Hom}_G(W, R^q\pi_*\omega_{X/S}(\xi)) \otimes W$ is a vector bundle on S. By definition,

$$\lambda_G(\omega_{X/S}(\xi)) = \prod_{W \in \widehat{G}} \bigotimes_{q \geq 0} \det(\mathrm{Hom}_G(W, R^q\pi_*\omega_{X/}(\xi)) \otimes W)^{(-1)^q}.$$

Let $r_W^q \in \mathbf{Z}_{\geq 0}$ be the rank of $\mathrm{Hom}_G(W, R^q\pi_*\Omega_X^{n+1}(\xi)) \otimes W$ as a free $\mathcal{O}_S$-module on S. Let $\{\Psi_1, \ldots, \Psi_{r_W^q}\} \subset H^0(S, \mathrm{Hom}_G(W, R^q\pi_*\Omega_X^{n+1}(\xi)) \otimes W)$ be a free basis of the locally free sheaf $\mathrm{Hom}_G(W, R^q\pi_*\Omega_X^{n+1}(\xi)) \otimes W$ on S. Define

$$\sigma_W^q := (\Psi_1 \otimes (\pi^*ds)^{-1}) \wedge \cdots \wedge (\Psi_{r_W^q} \otimes (\pi^*ds)^{-1})$$

if $\mathrm{Hom}_G(W, R^q\pi_*\omega_{X/S}(\xi)) \neq 0$. We set $\sigma_W^q := 1_{\lambda_W(\xi)}$ if $\mathrm{Hom}_G(W, R^q\pi_*\omega_{X/S}(\xi)) = 0$. Then σ_W^q generates $\det(\mathrm{Hom}_G(W, R^q\pi_*\omega_{X/S}(\xi)) \otimes W)$ on S.

6.2. **Semistable reduction.** Let T be the unit disc in $\mathbf{C}$. For $0 < \epsilon < 1$, we set $T(\epsilon) := \{t \in T; |t| < \epsilon\}$ and $T^o := T \setminus \{0\}$. By the semistable reduction theorem [17, Chap. II], there is a diagram

$$
\begin{array}{ccccc}
(Y, Y_0) & \xrightarrow{\ r\ } & (X \times_S T, X_0) & \xrightarrow{\ \mathrm{pr}_1\ } & (X, X_0) \\
f \downarrow & & \mathrm{pr}_2 \downarrow & & \pi \downarrow \\
(T, 0) & \xrightarrow{\ \mathrm{id}\ } & (T, 0) & \xrightarrow{\ \mu\ } & (S, 0)
\end{array}
$$

such that $Y_0 = f^{-1}(0)$ is a reduced normal crossing divisor. Here $\mu \colon (T, 0) \to (S, 0)$ is given by $\mu(t) = t^\nu$ for some $\nu \in \mathbf{Z}_{>0}$ and $r \colon Y \to X \times_S T$ is a projective resolution. Set $F = \mathrm{pr}_1 \circ r \colon Y \to X$. Since $(F^*\xi, h_{F^*\xi} := F^*h_\xi)$ is Nakano semi-positive, we may assume by Lemma 5.2 that $R^q f_*\Omega_Y^{n+1}(F^*\xi)$ is a free $\mathcal{O}_T$-module. Since $f|_{Y \setminus Y_0} \colon Y \setminus Y_0 \to T^o$ is G-equivariant, $R^q f_*\Omega_Y^{n+1}(F^*\xi)|_{T^o}$ is a G-equivariant holomorphic vector bundle on T^o.

Lemma 6.1. *The G-action on $R^q f_*\Omega_Y^{n+1}(F^*\xi)|_{T^o}$ extends to a holomorphic G-action on $R^q f_*\Omega_Y^{n+1}(F^*\xi)$.*

Proof. Let $\rho' \colon Z' \to X \times_S T$ be a G-equivariant resolution and set $\varpi' := \mathrm{pr}_2 \circ \rho'$ and $\Pi' := \mathrm{pr}_1 \circ \rho'$. By the G-equivariance of $\varpi' \colon Z' \to T$, $R^q\varpi'_*\Omega_{Z'}^{n+1}(\Pi'^*\xi)$ is a G-equivariant holomorphic vector bundle on T. Since Y is birational to Z', we get an

isomorphism $R^q f_* \Omega_Y^{n+1}(F^*\xi) \cong R^q \varpi'_* \Omega_{Z'}^{n+1}(\Pi'^*\xi)$ of holomorphic vector bundles on T by [28, Th. 6.9 (i)], which induces the desired G-action on $R^q f_* \Omega_Y^{n+1}(F^*\xi)$. $\square$

6.3. Estimate for the L^2-metric for the semistable family. We write t for the coordinate of $T \cong \Delta$ centered at 0. Set $\kappa_T = i\, dt \wedge d\bar{t}$, which is a Kähler form on T. By [23, Sect. 4.1],

$$\kappa_Y := F^*\kappa_{\mathcal{X}} + i\, f^*\kappa_T$$

is a C^∞ $(1,1)$-form on Y, which is a Kähler form only on $Y \setminus Y_0$. To get an estimate of the L^2-metric on $R^q f_* \omega_{Y/T}(F^*\xi)$ with respect to the degenerate Kähler form κ_Y, we need an analogue of Theorem 5.1 for the Nakano semi-positive vector bundle $(F^*\xi, F^*h_\xi)$ on (Y, κ_Y). Such an extension of Theorem 5.1 was given by Mourougane-Takayama [23, Prop. 4.4]. However, we can not apply it to our situation at one for the following reason: Set

$$\Sigma := \mathrm{Sing}(X \times_S T), \qquad U := (X \times_S T) \setminus \Sigma.$$

We may assume by [17, Chap. II] that $r \colon Y \setminus r^{-1}(\Sigma) \to U$ is an isomorphism and that $r^{-1}(\Sigma) \subset Y_0$ is a normal crossing divisor. However, it is not clear from the construction in [17, Chap. II] if $r \colon Y \to X \times_S T$ is a *composite of blowing-ups with non-singular centers* disjoint from U. Since this condition is essential in the construction of a sequence of Kähler forms on Y approximating κ_Y (cf. [23, Proof of Prop. 4.4 Step 1]), we can *not* apply the arguments in [23, Prop. 4.4] to the bundle $(F^*\xi, h_{F^*\xi})$ on (Y, κ_Y) at once. In stead of applying it to (Y, κ_Y), we apply it to a manifold Z dominating Y, whose construction is as follows.

By [4, Th. 13.2], there exists a resolution of the singularity $r' \colon W \to X \times_S T$, which is a composite of blowing-ups with non-singular centers disjoint from U. We apply [1, Lemma 1.3.1] by setting $X_1 = W$ and $X_2 = Y$. As a result, there exist a projective resolution $\rho \colon Z \to X \times_S T$ and a birational morphism $\varphi \colon Z \to Y$ such that ρ is a composite of blowing-ups with non-singular centers disjoint from U and $\rho = r \circ \varphi$. In particular, $Z \setminus \rho^{-1}(\Sigma) \cong Y \setminus r^{-1}(\Sigma) \cong U$ and $Z_0 = \varphi^{-1}(Y_0)$ is a normal crossing divisor . We set $\varpi := \mathrm{pr}_2 \circ \rho \colon Z \to T$ and $\Pi := \mathrm{pr}_1 \circ \rho \colon Z \to X$.

Regarding $X \times_S T$ as a hypersurface of $X \times T$, we get on Y (cf. [23, Sect. 4.1])

$$\kappa_Y = r^*(\kappa_X + \kappa_T).$$

Similarly, $\kappa_Z := \rho^*(\kappa_X + \kappa_T)$ is a degenerate Kähler form on Z with $\kappa_Z = \varphi^*\kappa_Y$. Since Z is obtained from $X \times_S T$ by a composite of blowing-ups with non-singular centers, we deduce from e.g. [12, Prop. 12.4], [23, Proof of Prop. 4.4 Step 1] the existence of a sequence of Kähler forms $\{\kappa_{Z,k}\}_{k \geq 1}$ on Z such that $\kappa_{Z,k} = \kappa_Z$ on $Z \setminus \varpi^{-1}(T(\frac{1}{k}))$. (In fact, we can assume $[\kappa_{Z,k}]|_{Z \setminus Z_0} = [\kappa_Z]|_{Z \setminus Z_0}$ for $k > 1$ by an appropriate construction of $\kappa_{Z,k}$. See Sect. 9.1 below.)

Proposition 6.2. *For every* $\Theta \in H^q(Y, \Omega_Y^{n+1}(F^*\xi))$, *there exists a holomorphic differential form* $\theta \in H^0(Y, \Omega_Y^{n+1-q}(F^*\xi))$ *such that*

$$\Theta|_{Y \setminus Y_0} = [\theta \wedge \kappa_Y^q]|_{Y \setminus Y_0} \in H^q(Y \setminus Y_0, \Omega_Y^{n+1}(F^*\xi)), \qquad \theta \wedge f^*dt = 0.$$

Proof. Since $\varphi^{-1}(Y) = Z$, we get $\varphi^*\Theta \in H^q(Z, \Omega_Z^{n+1}(\Pi^*\xi))$. By Theorem 5.1 applied to the Nakano semi-positive vector bundle $(\Pi^*\xi, \Pi^*h_\xi)$ on the Kähler manifold $(Z, \kappa_{Z,k})$, there exists a holomorphic differential form $\theta_k \in H^0(Z, \Omega_Z^{n+1-q}(\Pi^*\xi))$ such that

$$(6.1) \qquad \varphi^*\Theta = [\theta_k \wedge \kappa_{Z,k}^q], \qquad (\varpi^*ds) \wedge \theta_k = 0.$$

Let $\mathcal{W} \subset T$ be an open subset such that $\mathcal{W} \subset T \setminus T(\frac{1}{k})$ for all $k > 1$. Since $\kappa_{Z,k} = \kappa_Z$ on $Z \setminus \varpi^{-1}(T(\frac{1}{k}))$, we get $\kappa_{Z,k} = \kappa_Z$ on $\varpi^{-1}(\mathcal{W})$ for all $k > 1$. By [23, Proof of Prop. 4.4 Step 3], the equality $\kappa_{Z,k}|_{\varpi^{-1}(\mathcal{W})} = \kappa_{Z,l}|_{\varpi^{-1}(\mathcal{W})}$ implies that $\theta_k|_{\varpi^{-1}(\mathcal{W})} = \theta_l|_{\varpi^{-1}(\mathcal{W})}$ and hence $\theta_k = \theta_l$ for all $k, l > 1$. We set $\theta_\infty := \theta_k$. Then

$$(6.2) \qquad \varphi^*\Theta = [\theta_\infty \wedge \kappa_{Z,k}^q], \qquad\qquad (\varpi^*ds) \wedge \theta_\infty = 0$$

for all $k > 1$ by (6.1). Since $\kappa_{Z,k} = \kappa_Z$ on $Z \setminus \varpi^{-1}(T(\frac{1}{k}))$, we get the equality of cohomology classes

$$(6.3) \qquad \varphi^*\Theta|_{Z \setminus \varpi^{-1}(T(\frac{1}{k}))} = [\theta_\infty \wedge \kappa_Z^q]|_{Z \setminus \varpi^{-1}(T(\frac{1}{k}))}$$

for all $k > 1$. Since $k > 1$ is arbitrary, we get by (6.3)

$$(6.4) \qquad \varphi^*\Theta|_{Z \setminus Z_0} = [\theta_\infty \wedge \kappa_Z^q]|_{Z \setminus Z_0}, \qquad Z_0 := \varpi^{-1}(0).$$

(In fact, $\varphi^*\Theta = [\theta_\infty \wedge \kappa_Z^q]$. See Sect. 9.2 below.)

Since $\varphi \colon Z \to Y$ induces an isomorphism between $Z \setminus \rho^{-1}(\Sigma)$ and $Y \setminus r^{-1}(\Sigma)$, we get $(\varphi^{-1})^*\theta_\infty \in H^0(Y \setminus Y_0, \Omega_Y^{n+1-q}(F^*\xi))$. Let ω_Y be a Kähler form on Y. Since $\theta_\infty \wedge \overline{\theta_\infty} \wedge \varphi^*\omega_Y^q$ is a C^∞ top form on Z, we get $(\varphi^{-1})^*\theta_\infty \in L^2_{\mathrm{loc}}(Y)$. In particular, $(\varphi^{-1})^*\theta_\infty$ extends to a holomorphic differential form on Y. Hence there exists $\theta \in H^0(Y, \Omega_Y^{n+1-q}(F^*\xi))$ such that $(\varphi^{-1})^*\theta_\infty = \theta|_{Y \setminus Y_0}$. Since $\kappa_Z = \varphi^*\kappa_Y$, we get

$$\Theta|_{Y \setminus Y_0} = (\varphi^{-1})^*\varphi^*\Theta|_{Z \setminus Z_0} = (\varphi^{-1})^*[\theta_\infty \wedge \kappa_Z^q]|_{Z \setminus Z_0} = [\theta \wedge \kappa_Y^q]|_{Y \setminus Y_0}.$$

Since $\varpi = f \circ \varphi$ and hence $(\varphi^{-1})^*\varpi^* = f^*$, we get $(f^*ds) \wedge \theta = 0$ by the relation $(\varpi^*ds) \wedge \theta_\infty = 0$. This completes the proof. $\qquad\square$

In fact, we get $\Theta = [\theta \wedge \kappa_Y^q]$ in Proposition 6.2. See Sect. 9.2.

Let $\{\Theta_1, \dots, \Theta_{r_W^q}\}$ be a basis of $\mathrm{Hom}_G(W, R^q f_* \Omega_Y^{n+1}(F^*\xi)) \otimes W$ as a free $\mathcal{O}_T$-module. Shrinking T if necessary, we may assume $\Theta_\alpha \in H^q(Y, \Omega_Y^{n+1}(F^*\xi))$. By Proposition 6.2, there exist holomorphic differential forms $\theta_\alpha \in H^0(Y, \Omega_Y^{n-q+1}(F^*\xi))$ and $\Xi_\alpha \in H^0(Y \setminus Y_0, \Omega_{Y/T}^{n-q}(F^*\xi))$ such that

$$(6.5) \qquad \Theta_\alpha|_{Y \setminus Y_0} = [\theta_\alpha \wedge \kappa_Y^q]|_{Y \setminus Y_0} \in H^q(Y \setminus Y_0, \Omega_Y^{n+1}(F^*\xi)),$$

$$(6.6) \qquad\qquad\qquad \theta_\alpha|_{Y \setminus Y_0} = \Xi_\alpha \wedge f^*dt.$$

For $t \in T^o$, we set

$$H_{\alpha\bar{\beta}}(t) := (\Theta_\alpha \otimes (f^*dt)^{-1}|_{Y_t}, \Theta_\beta \otimes (f^*dt)^{-1}|_{Y_t})_{L^2}.$$

Then $H(t) = (H_{\alpha\bar{\beta}}(t))$ is a positive-definite $r_W^q \times r_W^q$-Hermitian matrix. Set $\kappa_{Y_t} := \kappa_Y|_{Y_t}$ for $t \in T^o$. Since $\Xi_\alpha \wedge \kappa_Y^q|_{Y_t}$ is the harmonic representative of its class $[\Xi_\alpha \wedge \kappa_Y^q]|_{Y_t}$ with respect to the metrics κ_{Y_t}, $h_{F^*\xi}|_{Y_t}$, we get for all $t \in T^o$

$$H_{\alpha\bar{\beta}}(t) = ([\Xi_\alpha \wedge \kappa_Y^q]|_{Y_t}, [\Xi_\beta \wedge \kappa_Y^q]|_{Y_t})_{L^2} = \int_{Y_t} i^{(n-q)^2} h_{F^*\xi}(\Xi_\alpha \wedge \overline{\Xi}_\beta)|_{Y_t} \wedge \kappa_{Y_t}^q.$$

Here $h_{F^*\xi}(\Xi_\alpha \wedge \overline{\Xi}_\beta)$ is defined as follows: Let $\{e_1, \dots, e_r\}$ be a local frame of $F^*\xi$ and let $\{e_1^\vee, \dots, e_r^\vee\}$ be its dual frame of $F^*\xi^\vee$. We can express locally $\Xi_\alpha = \sum_i \Xi_{\alpha,i} \otimes e_i$ and $h_{F^*\xi} = \sum_{i,j} h_{i\bar{j}} e_i^\vee \otimes \bar{e}_j^\vee$, where $\Xi_{\alpha,i}$ is a local holomorphic section of $\Omega_{Y/T}^q$ and $h_{i\bar{j}}$ is a local C^∞ function. We define

$$h_{F^*\xi}(\Xi_\alpha \wedge \overline{\Xi}_\beta) := \sum_{i,j} h_{i\bar{j}} \Xi_{\alpha,i} \wedge \overline{\Xi}_{\beta,j} \in C^\infty(Y \setminus Y_0, \Omega_{Y/T}^q \wedge \overline{\Omega}_{Y/T}^q).$$

Then $h_{F^*\xi}(\Xi_\alpha \wedge \overline{\Xi}_\beta)|_{Y_t} = h_{F^*\xi}(\Xi_\alpha|_{Y_t} \wedge \overline{\Xi_\beta|_{Y_t}}) \in A_{Y_t}^{q;q}$ for all $t \in T^o$.

Lemma 6.3. *One has $H_{\alpha\bar\beta}(t) \in \bigoplus_{m=0}^{n}(\log|t|)^m C^\infty(T)$, so that there exist constants $a_{\alpha\bar\beta;m} \in \mathbf{C}$ with*

$$H_{\alpha\bar\beta}(t) = \sum_{m=0}^{n} a_{\alpha\bar\beta;m} \left(\log|t|^2\right)^m + O\left(|t|(\log|t|)^n\right) \qquad (t \to 0).$$

In particular, $\det H(t) \in \bigoplus_{m=0}^{n}(\log|t|)^m C^\infty(T)$ and there exist constants $c_m \in \mathbf{R}$, $0 \le m \le nr_W^q$ with

$$\det H(t) = \sum_{m=0}^{nr_W^q} c_m \left(\log|t|^2\right)^m + O\left(|t|(\log|t|)^{nr_W^q}\right) \qquad (t \to 0).$$

Proof. By (6.6), we get on $Y \setminus Y_0$

(6.7)
$$f^*(i\,dt \wedge d\bar{t}) \wedge \{i^{(n-q)^2} h_{F^*\xi}(\Xi_\alpha \wedge \overline{\Xi}_\beta) \wedge \kappa_Y^q\} = i^{(n-q+1)^2} h_{F^*\xi}(\theta_\alpha \wedge \overline{\theta}_\beta) \wedge \kappa_Y^q.$$

Let $\{\mathcal{V}_\lambda\}_{\lambda \in \Lambda}$ be an open covering of Y with $\#\Lambda < +\infty$ such that there is a system of coordinates $(z_0, \ldots, z_n)$ on V_λ with $f|_{\mathcal{V}_\lambda}(z) = z_0 \cdots z_k$. Here k depends on $\lambda \in \Lambda$. Let $\{\varrho_\lambda\}_{\lambda \in \Lambda}$ be a partition of unity subject to the covering $\{\mathcal{V}_\lambda\}_{\lambda \in \Lambda}$. On $\mathcal{V}_\lambda$, we define

$$\tau := \frac{1}{k}\sum_{j=0}^{k}(-1)^{j-1} z_j\, dz_0 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_k \wedge dz_{k+1} \wedge \cdots \wedge dz_n.$$

Then $\pi^*(dt/t) \wedge \tau = dz_0 \wedge \cdots \wedge dz_n$ on $\mathcal{V}_\lambda$. Since θ_α and θ_β are holomorphic $n-q+1$-forms on Y and since $\kappa_Y \in A_Y^{1;1}$, there exists $B_{\alpha\bar\beta}(z) \in C_0^\infty(\mathcal{V}_\lambda)$ such that

(6.8)
$$\varrho_\lambda(z) h_{F^*\xi}(\theta_\alpha \wedge \overline{\theta}_\beta) \wedge \kappa_Y^{n-q}|_{\mathcal{V}_\lambda} = B_{\alpha\bar\beta}(z)\, dz_0 \wedge \cdots \wedge dz_n \wedge \overline{dz_0 \wedge \cdots \wedge dz_n}$$
$$= (-1)^n B_{\alpha\bar\beta}(z) \left(\frac{\tau}{f^*t}\right) \wedge \overline{\left(\frac{\tau}{f^*t}\right)} \wedge f^*(dt \wedge d\bar{t}).$$

Comparing (6.7) and (6.8), we get

(6.9)
$$\int_{Y_t \cap \mathcal{V}_\lambda} i^{(n-q)^2} \varrho_\lambda(z) h_{F^*\xi}(\Xi_\alpha \wedge \overline{\Xi}_\beta)|_{Y_t} \wedge \kappa_{Y_t}^q = |t|^{-2} \int_{Y_t \cap \mathcal{V}_\lambda} i^{(n-q)^2} B_{\alpha\bar\beta}(z)\, \tau \wedge \overline{\tau}.$$

Since $\tau/f^*t = \frac{1}{k}\sum_{j=0}^{k}(-1)^{j-1}\frac{dz_0}{z_0} \wedge \cdots \wedge \frac{dz_{j-1}}{z_{j-1}} \wedge \frac{dz_{j+1}}{z_{j+1}} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge dz_{k+1} \wedge \cdots \wedge dz_n$ has only logarithmic singularities, there exists by (6.9) a constant $C_0 > 0$ such that

(6.10)
$$\left|\int_{Y_t \cap \mathcal{V}_\lambda} \varrho_\lambda(z) h_{F^*\xi}(\Xi_\alpha \wedge \overline{\Xi}_\beta)|_{Y_t} \wedge \kappa_{Y_t}^q\right| \le C_0 \left(-\log|t|^2\right)^k$$

for all $t \in T^o$. On the other hand, we deduce from [2, p.166 Th. 4bis.] that $f_*(B_{\alpha\bar\beta}\,\tau \wedge \overline{\tau}) \in C^\infty(T) \oplus \bigoplus_{m=0}^{n} |t|^2(\log|t|)^m C^\infty(T)$ is of the following form

(6.11)
$$\int_{Y_t \cap \mathcal{V}_\lambda} i^{(n-q)^2} B_{\alpha\bar\beta}(z)\, \tau \wedge \overline{\tau} = a_{\alpha\bar\beta}^{(\lambda)} + b_{\alpha\bar\beta}^{(\lambda)}\, t + c_{\alpha\bar\beta}^{(\lambda)}\, \bar{t} + \sum_{m=0}^{k} a_{\alpha\bar\beta;m}^{(\lambda)}(t)\, |t|^2(\log|t|^2)^m$$

as $t \to 0$, where $a_{\alpha\bar{\beta}}^{(\lambda)}$, $b_{\alpha\bar{\beta}}^{(\lambda)}$, $c_{\alpha\bar{\beta}}^{(\lambda)}$ are constants and $a_{\alpha\bar{\beta};m}^{(\lambda)}(t) \in C^\infty(T)$. Comparing (6.9), (6.10), (6.11), we get $a_{\alpha\bar{\beta}}^{(\lambda)} = b_{\alpha\bar{\beta}}^{(\lambda)} = c_{\alpha\bar{\beta}}^{(\lambda)} = 0$ and

(6.12)
$$\int_{Y_t \cap \mathcal{V}_\lambda} i^{(n-q)^2} \varrho_\lambda(z) h_{F^*\xi} (\Xi_\alpha \wedge \overline{\Xi}_\beta)|_{Y_t} \wedge \kappa_{Y_t}^q = \sum_{m=0}^k a_{\alpha\bar{\beta};m}^{(\lambda)}(t)\, (\log|t|^2)^m.$$

Since $f_*(\varrho_\lambda h_{F^*\xi}(\Xi_\alpha \wedge \overline{\Xi}_\beta) \wedge \kappa_Y^q) \in \bigoplus_{m=0}^n (\log|t|)^m\, C^\infty(T)$ by (6.12), we get

$$H_{\alpha\bar{\beta}}(t) = \sum_{\lambda \in \Lambda} i^{(n-q)^2} f_*(\varrho_\lambda\, h_{F^*\xi}(\Xi_\alpha \wedge \overline{\Xi}_\beta) \wedge \kappa_Y^q) \in \bigoplus_{m=0}^n (\log|t|)^m\, C^\infty(T).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 6.4. *There exists $C > 0$ such that the following inequality holds for all $u = (u_1, \ldots, u_{r_W^q}) \in \mathbf{C}^{r_W^q}$ and $t \in T^o$:*

$$\Big\| \sum_\alpha u_\alpha \Theta_\alpha \otimes (f^*dt)^{-1}|_{Y_t} \Big\|_{L^2}^2 \geq C\,|u|^2.$$

In particular, the following inequality holds for all $t \in T^o$

$$\det H(t) \geq C^{r_W^q}.$$

Proof. By [28, Th. 6.9 (i)], $\{(\varphi^*\Theta_1) \otimes (\varpi^*dt)^{-1}, \ldots, (\varphi^*\Theta_{r_W^q}) \otimes (\varpi^*dt)^{-1}\}$ is a basis of the free $\mathcal{O}_T$-module $\mathrm{Hom}_G(W, R^q\varpi_*\omega_{Z/T}(\varphi^*F^*\xi)) \otimes W$. Since $\Theta_\alpha|_{Y\backslash Y_0} = [\Xi_\alpha \wedge \kappa_Y^q \wedge (f^*dt)]$ by (6.5), (6.6), we get $\varphi^*\Theta_\alpha|_{Z\backslash Z_0} = [(\varphi^*\Xi_\alpha) \wedge \kappa_Z^q \wedge (\varpi^*dt)]$.

Since $\rho \colon Z \to X \times_T S$ is a composite of blowing-ups with non-singular centers disjoint from U and hence satisfies [23, Sect. 2.3 (1), (2)], there is a constant $C_0 > 0$ by [23, Lemmas 4.7 and 4.8] such that for all $u = (u_\alpha) \in \mathbf{C}^{r_W^q}$ and $t \in T^o$,

$$\Big\| \sum_\alpha u_\alpha(\varphi^*\Xi_\alpha) \wedge \kappa_Z^q|_{Z_t} \Big\|_{L^2}^2 = \Big\| \sum_\alpha u_\alpha \varphi^*\Xi_\alpha|_{Z_t} \Big\|_{L^2}^2 \geq C_0\,|u|^2.$$

Since $\kappa_Z = \varphi^*\kappa_Y$ and hence

$$\Big\| \sum_\alpha u_\alpha \Theta_\alpha \otimes (f^*dt)^{-1}|_{Y_t} \Big\|_{L^2}^2 = \Big\| \sum_\alpha u_\alpha \Xi_\alpha \wedge \kappa_Y^q|_{Y_t} \Big\|_{L^2}^2 = \Big\| \sum_\alpha u_\alpha(\varphi^*\Xi_\alpha) \wedge \kappa_Z^q|_{Z_t} \Big\|_{L^2}^2,$$

we get the first inequality. Since ${}^t v H(t)\bar{v} \geq C_0\|v\|^2$ for all $v \in \mathbf{C}^{r_W^q}$ and $t \in T^o$ by the first inequality, the smallest eigenvalue of $H(t)$ is greater than or equal to $C_0 > 0$, which implies $\det H(t) \geq C_0^{r_W^q}$. $\qquad\qquad\qquad\square$

Proposition 6.5. *There exist an integer $\nu_W^q \in \mathbf{Z}_{\geq 0}$ and a constant $c_W^q \in \mathbf{R}$ such that as $t \to 0$,*

$$\log \det H(t) = \nu_W^q \log\left(-\log|t|^2\right) + c_W^q + O\left(1/\log|t|\right).$$

Proof. By the second statement of Lemma 6.4, there exists non-zero c_m in Lemma 6.3. Let $c_N(\log|t|)^N$ be the leading term of the expansion of $\det H(t)$. Namely, $c_N \neq 0$ and $c_m = 0$ for $m > N$ in Lemma 6.3. Then we get by Lemma 6.3

$$\det H(t) = c_N(\log|t|)^N \left(1 + O\left(1/\log|t|\right)\right) \qquad (t \to 0).$$

The result follows from this estimate. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 6.6. *Let $\varsigma \in \Gamma(T, R^q f_* \omega_{Y/T}(F^*\xi))$. Then $R^q f_* \omega_{Y/T}(F^*\xi)/\mathcal{O}_T\varsigma$ is a free $\mathcal{O}_T$-module near $t = 0$ if and only if*

$$\log \|\varsigma(t)\|_{L^2} = O\left(\log(-\log|t|)\right) \qquad (t \to 0).$$

In particular, $\tau \in \Gamma(T^o, R^q f_* \omega_{Y/T}(F^*\xi))$ extends to a holomorphic section defined on T if and only if there exists $e \in \mathbf{Z}_{\geq 0}$ with

$$\log \|\tau(t)\|_{L^2} = e \log|t| + O\left(\log(-\log|t|)\right) \qquad (t \to 0).$$

Proof. Set $l_q := h^q(Y_t, \omega_{Y_t}(F^*\xi))$. If $R^q f_* \omega_{Y/T}(F^*\xi)/\mathcal{O}_T\varsigma$ is free around $t = 0$, then there is a basis $\{\Theta_1, \ldots, \Theta_{l_q}\}$ of $R^q f_* \Omega_Y^{n+1}(F^*\xi)$ around $t = 0$ with $\varsigma = \Theta_1 \otimes (f^*dt)^{-1}$. The desired estimate $C \leq \|\varsigma(t)\|_{L^2} \leq C(-\log|t|)^{nl_q}$ for all $t \in T^o$ follows from (6.11) and Lemma 6.4 after shrinking T if necessary.

Assume $\log \|\varsigma(t)\|_{L^2} = O\left(\log(-\log|t|)\right)$ as $t \to 0$. Let $\{\Theta_1, \ldots, \Theta_{l_q}\}$ be a basis of $R^q f_* \Omega_Y^{n+1}(F^*\xi)$ on $T(\epsilon)$ for some $\epsilon > 0$. On $T(\epsilon)$, we can express $\varsigma(t) = \sum_\alpha c_\alpha(t) \Theta_\alpha \otimes (f^*dt)^{-1}|_{Y_t}$, where $c_\alpha(t) \in \mathcal{O}(T(\epsilon))$. Let $\nu \in \mathbf{Z}_{\geq 0}$ be such that $\varsigma(t)/t^\nu \in \mathcal{O}(T(\epsilon))$ and $\varsigma(t)/t^{\nu+1} \notin \mathcal{O}(T(\epsilon))$. There is a basis $\{\Theta_1', \ldots, \Theta_{l_q}'\}$ of $R^q f_* \Omega_Y^{n+1}(F^*\xi)$ with $\Theta_1' \otimes (f^*dt)^{-1} = t^{-\nu}\varsigma(t)$. Since $C \leq \|\Theta_1' \otimes (f^*dt)^{-1}|_{Y_t}\|_{L^2} \leq C(-\log|t|)^{nl_q}$ by (6.11) and Lemma 6.4, $\log \|\varsigma(t)\|_{L^2} = -\nu \log|t| + O(\log(-\log|t|))$ as $t \to 0$. Since $\log \|\varsigma(t)\|_{L^2} = O\left(\log(-\log|t|)\right)$ as $t \to 0$ by assumption, we get $\nu = 0$. Hence $\varsigma = \Theta_1' \otimes (f^*dt)^{-1}$ is a part of a basis of $R^q f_* \Omega_Y^{n+1}(F^*\xi)$. $\qquad\square$

6.4. Comparison of the L^2-metrics.

We recall the following result of Mourougane-Takayama.

Proposition 6.7. *There is a natural injection*

$$(6.13) \qquad \varphi \colon R^q f_* \omega_{Y/T}(F^*\xi) \hookrightarrow \mu^* R^q \pi_* \omega_{X/S}(\xi)$$

with the following properties:

(1) *$\mu^* R^q \pi_* \omega_{X/S}(\xi)/\varphi(R^q f_* \omega_{Y/T}(F^*\xi))$ is a torsion sheaf on T supported at 0.*
(2) *φ preserves the L^2-metrics, i.e.,*

$$\varphi^* \mu^* h_{R^q \pi_* \omega_{X/S}(\xi)} = h_{R^q f_* \omega_{Y/T}(F^*\xi)}.$$

Here $h_{R^q \pi_* \omega_{X/S}(\xi)}$ (resp. $h_{R^q f_* \omega_{Y/T}(F^*\xi)}$) is the L^2-metric on $R^q \pi_* \omega_{X/S}(\xi)$ (resp. $R^q f_* \omega_{Y/T}(F^*\xi)$) with respect to κ_X, h_ξ (resp. κ_Y, F^*h_ξ).

Proof. See [23, Lemmas 3.3 and 4.2]. $\qquad\square$

We remark that since $\varphi|_{T^o}$ is G-equivariant and φ is defined on T, φ is G-equivariant on T.

Let $\mathfrak{m}_0 = t\,\mathcal{O}_T \subset \mathcal{O}_T$ be the ideal sheaf of $0 \in T$. For $q \geq 0$ and $W \in \widehat{G}$, we set

$$(6.14) \qquad \delta_W^q := \dim_{\mathcal{O}_T/\mathfrak{m}_0} \frac{\mathrm{Hom}_G(W, \mu^* R^q \pi_* \omega_{X/S}(\xi)) \otimes W}{\mathrm{Hom}_G(W, R^q f_* \omega_{Y/T}(F^*\xi)) \otimes W} \in \mathbf{Z}_{\geq 0}.$$

Theorem 6.8. *By choosing the basis $\{\Psi_1, \ldots, \Psi_{r_W^q}\}$ and $\{\Theta_1, \ldots, \Theta_{r_W^q}\}$ appropriately, there exist integers $e_1, \ldots, e_{r_W^q} \geq 0$ such that the $r_W^q \times r_W^q$-Hermitian matrix $G(s) := ((\Psi_\alpha \otimes (\pi^*ds)^{-1}|_{X_s}, \Psi_\beta \otimes (\pi^*ds)^{-1}|_{X_s})_{L^2})$ is expressed as follows:*

$$G(\mu(t)) = D(t) \cdot H(t) \cdot \overline{D(t)}, \qquad D(t) = \mathrm{diag}(t^{-e_1}, \ldots, t^{-e_{\rho_q}}).$$

In particular, as $s \to 0$,

$$\log \|\sigma_W^q(s)\|_{L^2}^2 = -\frac{\delta_W^q}{\deg \mu} \log|s|^2 + \nu_W^q \log\left(-\log|s|^2\right) + c_W^q + O\left(1/\log|s|\right).$$

Proof. By choosing the basis $\{\Psi_\alpha\}$ and $\{\Theta_\alpha\}$ suitably, there exists by Proposition 6.7 (1) an integer $e_\alpha \in \mathbf{Z}_{\geq 0}$ for all $1 \leq \alpha \leq r_W^q$ such that the following equality holds on T:

$$\mu^*(\Psi_\alpha \otimes (\pi^* dt)^{-1})|_{Y_t} = t^{-e_\alpha} \varphi(\Theta_\alpha \otimes (f^* dt)^{-1}|_{Y_t}).$$

Then we get

(6.15)
$$
\begin{aligned}
G_{\alpha\bar\beta}(\mu(t)) &= (\Psi_\alpha \otimes (\pi^* ds)^{-1}|_{X_{t^\nu}}, \Psi_\beta \otimes (\pi^* ds)^{-1}|_{X_{t^\nu}})_{L^2} \\
&= h_{R^q \pi_* \omega_{X/S}(\xi)}(\Psi_\alpha \otimes (\pi^* ds)^{-1}, \Psi_\beta \otimes (\pi^* ds)^{-1})(\mu(t)) \\
&= t^{-e_\alpha} \bar t^{-e_\beta} h_{R^q \pi_* \omega_{X/S}(\xi)}(\varphi(\Theta_\alpha \otimes (f^* dt)^{-1}), \varphi(\Theta_\beta \otimes (f^* dt)^{-1}))(\mu(t)) \\
&= t^{-e_\alpha} \bar t^{-e_\beta} \varphi^* \mu^* h_{R^q \pi_* \omega_{X/S}(\xi)}(\Theta_\alpha \otimes (f^* dt)^{-1}, \Theta_\beta \otimes (f^* dt)^{-1})(t) \\
&= t^{-e_\alpha} \bar t^{-e_\beta} h_{R^q f_* \omega_{Y/T}(F^*\xi)}(\Theta_\alpha \otimes (f^* dt)^{-1}, \Theta_\beta \otimes (f^* dt)^{-1})(t) \\
&= t^{-e_\alpha} \bar t^{-e_\beta} H_{\alpha\bar\beta}(t),
\end{aligned}
$$

where the fifth equality follows from Proposition 6.7 (2). This proves the first equality of Proposition 6.8. Since

(6.16)
$$
\begin{aligned}
\sum_\alpha e_\alpha &= \dim_{\mathcal{O}_T/\mathfrak{m}_0} \bigoplus_\alpha \mathcal{O}_T \Psi_\alpha \otimes (\pi^* ds)^{-1}/\mathcal{O}_T \varphi(\Theta_\alpha \otimes (f^* dt)^{-1}) \\
&= \dim_{\mathcal{O}_T/\mathfrak{m}_0} \frac{\mathrm{Hom}_G(W, \mu^* R^q \pi_* \omega_{X/S}(\xi)) \otimes W}{\mathrm{Hom}_G(W, R^q f_* \omega_{Y/S}(F^*\xi)) \otimes W} = \delta_W^q,
\end{aligned}
$$

we get by (6.15), (6.16)

(6.17)
$$\det G(\mu(t)) = |t|^{-2\delta_W^q} \det H(t).$$

Since $|t| = |s|^{\frac{1}{\nu}}$ and $\|\sigma_W^q(s)\|_{L^2}^2 = \det G(s)$, the second equality of Proposition 6.8 follows from (6.17) and Proposition 6.5. $\qquad\square$

We remark that one can get Theorem 6.8 by using the method of variation of Hodge structures when (ξ, h_ξ) is a trivial Hermtian line bundle on $\mathcal{X}$ [34, Sect. 2.2]. For an application of Theorem 6.8 to the curvature of L^2-metric, see [34].

We define
$$\mathbb{L}_g\left(\frac{\mu^* R\pi_* \omega_{X/S}(\xi)}{Rf_* \omega_{Y/T}(F^*\xi)}\right) := \sum_{q \geq 0} (-1)^q \mathrm{Tr}[g|_{(\mu^* R^q \pi_* \omega_{X/S}(\xi)/R^q f_* \omega_{Y/T}(F^*\xi))_0}].$$

Lemma 6.9. *The following identity holds*

$$\mathbb{L}_g(\mu^* R\pi_* \omega_{X/S}(\xi)/Rf_* \omega_{Y/T}(F^*\xi)) = \sum_{W \in \widehat{G}} \sum_{q \geq 0} (-1)^q \frac{\chi_W(g)}{\dim W} \delta_W^q.$$

Proof. Since

$$\frac{\mu^* R^q \pi_* \omega_{X/S}(\xi)}{R^q f_* \omega_{Y/T}(F^*\xi)} = \bigoplus_{W \in \widehat{G}} \frac{\mathrm{Hom}_G(W, \mu^* R^q \pi_* \omega_{X/S}(\xi)) \otimes W}{\mathrm{Hom}_G(W, R^q f_* \omega_{Y/T}(F^*\xi)) \otimes W},$$

we get

(6.18)
$$\operatorname{Tr}\left[g|_{(\mu^* R^q \pi_* \omega_{X/S}(\xi)/R^q f_* \omega_{Y/T}(F^*\xi))_0}\right]$$

$$= \sum_{W \in \widehat{G}} \left(\dim_{\mathcal{O}_T/\mathfrak{m}_0} \frac{\operatorname{Hom}_G(W, \mu^* R^q \pi_* \omega_{X/S}(\xi))}{\operatorname{Hom}_G(W, R^q f_* \omega_{Y/T}(F^*\xi))}\right) \operatorname{Tr}[g|_W]$$

$$= \sum_{W \in \widehat{G}} \left(\dim_{\mathcal{O}_T/\mathfrak{m}_0} \frac{\operatorname{Hom}_G(W, \mu^* R^q \pi_* \omega_{X/S}(\xi))}{\operatorname{Hom}_G(W, R^q f_* \omega_{Y/T}(F^*\xi))}\right) \chi_W(g)$$

$$= \sum_{W \in \widehat{G}} \left(\dim_{\mathcal{O}_T/\mathfrak{m}_0} \frac{\operatorname{Hom}_G(W, \mu^* R^q \pi_* \omega_{X/S}(\xi)) \otimes W}{\operatorname{Hom}_G(W, R^q f_* \omega_{Y/T}(F^*\xi)) \otimes W}\right) \frac{\chi_W(g)}{\dim W} = \sum_{W \in \widehat{G}} \delta_W^q \frac{\chi_W(g)}{\dim W},$$

from which the result follows. $\qquad\square$

6.5. **Proof of Theorem 1.1.** By the definition of σ_W, $\sigma := (\sigma_W)_{W \in \widehat{G}}$ is an admissible section of $\lambda_G(\omega_{X/S}(\xi)) = \lambda_G(\omega_{\mathcal{X}/C}(\xi))|_S$. By Theorem 4.4, we get

(6.19)
$$\log \|\sigma(s)\|^2_{\lambda_G(\omega_{\mathcal{X}/C}(\xi)), Q}(g) = \alpha_g(X_0, \omega_{X/S}(\xi)) \log |s|^2 + O(1)$$

as $s \to 0$. On the other hand, we get by Proposition 6.8 and Lemma 6.9

(6.20)
$$\log \|\sigma(s)\|^2_{\lambda_G(\omega_{\mathcal{X}/C}(\xi)), Q}(g)$$

$$= \log \tau_G(X_s, \omega_{X_s}(\xi_s))(g) + \sum_{q \geq 0, W \in \widehat{G}} (-1)^q \frac{\chi_W(g)}{\dim W} \log \|\sigma_W^q(s)\|^2_{L^2}$$

$$= \log \tau_G(X_s, \omega_{X_s}(\xi_s))(g) +$$

$$\sum_{q, W} (-1)^q \frac{\chi_W(g)}{\dim W} \left\{ -\frac{\delta_W^q}{\deg \mu} \log |s|^2 + \nu_W^q \log(-\log |s|^2) + c_W^q + O\left(\frac{1}{\log |s|}\right) \right\}$$

$$= \log \tau_G(X_s, \omega_{X_s}(\xi_s))(g) - \frac{1}{\deg \mu} \mathbb{L}_g(\mu^* R\pi_* \omega_{X/S}(\xi)/Rf_* \omega_{Y/T}(F^*\xi)) \log |s|^2$$

$$+ \left(\sum_{q, W} (-1)^q \frac{\chi_W(g)}{\dim W} \nu_W^q\right) \log(-\log |s|^2) + \sum_{q, W} (-1)^q \frac{\chi_W(g)}{\dim W} c_W^q + O\left(\frac{1}{\log |s|}\right).$$

Comparing (6.19) and (6.20), we get

(6.21)
$$\log \tau_G(X_s, \omega_{X_s}(\xi_s))(g) = \beta_g \log |s|^2 + \nu_g \log(-\log |s|^2) + c_g + O(1/\log |s|)$$

as $s \to 0$, where we used the following notation in (6.21)

(6.22)
$$\begin{aligned}
\beta_g &:= \alpha_g(X_0, \omega_{X/S}(\xi)) + \frac{1}{\deg \mu} \mathbb{L}_g\left(\mu^* R\pi_* \omega_{X/S}(\xi)/Rf_* \omega_{Y/T}(F^*\xi)\right), \\
\nu_g &:= \sum_{q \geq 0, W \in \widehat{G}} (-1)^{q+1} \chi_W(g) \nu_W^q / \dim W, \\
c_g &:= \sum_{q \geq 0, W \in \widehat{G}} (-1)^{q+1} \chi_W(g) c_W^q / \dim W.
\end{aligned}$$

This completes the proof. $\qquad\square$

Corollary 6.10. *Let (E, h_E) be a holomorphic Hermitian vector bundle on $\mathcal{X}$. If (E, h_E) is semi-negative in the dual Nakano sense on X, then as $s \to 0$*

$$(-1)^{n+1} \log \tau_G(X_s, E_s)(g) = \beta_g \log |s|^2 + \nu_g \log(-\log |s|^2) + c_g + O(1/\log |s|).$$

Here β_g, ν_g, c_g are constants defined by the formula (6.22) by setting $\xi = E^\vee$.

Proof. Let $\square_{E_s}^{p,q}$ denote the Laplacian acting on $A_{X_s}^{p,q}(\xi)$ and let $*$ be the Hodge star operator. Since $*\square_{E_s}^{0,q}*^{-1} = \square_{E_s^\vee}^{n,n-q} = \square_{\Omega_{X_s}^n(E_s^\vee)}^{0,n-q}$, we get the relation

$$(6.23) \qquad \log \tau_G(X_s, E_s)(g) = (-1)^{n+1} \log \tau_G(X_s, \omega_{X_s}(E_s^\vee))(g).$$

Since $(E^\vee, h_{E^\vee})$ is Nakano semi-positive on X, the result follows from Theorem 1.1 and (6.23). $\qquad\square$

Corollary 6.11. *If (ξ, h_ξ) is Nakano semi-positive on X and if X_0 is a reduced normal crossing divisor of X, then as $s \to 0$*

$$\log \tau_G(X_s, \omega_{X_s}(\xi_s))(g) = \alpha_g(X_0, \omega_{X/S}(\xi)) \log |s|^2 + \nu_g \log(-\log |s|^2) + c_g + O\left(\frac{1}{\log |s|}\right).$$

Proof. We get $\mathbb{L}_g(\mu^* R\pi_* \omega_{X/S}(\xi)/Rf_* \omega_{Y/T}(F^*\xi)) = 0$ in Theorem 1.1, since $\pi \colon X \to S$ is a semistable degeneration. The result follows from Theorem 1.1. $\qquad\square$

7. (Log-)Canonical singularities and analytic torsion

7.1. (Log-)Canonical singularities. Let V be an n-dimensional normal projective variety with *locally free* dualizing sheaf ω_V. Set $V_{\mathrm{reg}} := V \setminus \mathrm{Sing}\, V$ and let $i \colon V_{\mathrm{reg}} \hookrightarrow V$ be the inclusion. Then $\omega_V = i_* \Omega_{V_{\mathrm{reg}}}^n$. The zero divisor of a holomorphic section of ω_V is called a canonical divisor of V and is denoted by K_V. Then V has only *canonical* (resp. *log-canonical*) singularities if there exist a resolution $\varphi \colon \widetilde{V} \to V$ and an φ-exceptional normal crossing divisor $E = \sum_{i \in I} a_i E_i \subset \widetilde{V}$ such that $K_{\widetilde{V}} = \varphi^* K_V + E$ and $a_i \geq 0$ (resp. $a_i \geq -1$) for all $i \in I$. Here E_i are irreducible and reduced divisors of $\widetilde{V}$. If V has only canonical singularities, then $\varphi^* \colon H^0(V, \omega_V) \to H^0(\widetilde{V}, \omega_{\widetilde{V}})$ is an isomorphism. In particular, every element of $H^0(V, \omega_V)$ is square integrable with respect to any Hermitian metric on V_{reg}.

Let W be a smooth projective manifold and let $D \subset W$ be a divisor. A birational morphism $\varphi \colon \widetilde{W} \to W$ between smooth projective manifolds is called an embedded resolution of D if φ is an isomorphism between $\widetilde{W} \setminus \varphi^{-1}(\mathrm{Sing}\, D)$ and $W \setminus \mathrm{Sing}\, D$ such that $\widetilde{D}$, the proper transform of D, is smooth. The pair (W, D) has only canonical (resp. log-canonical) singularities if there exist an embedded resolution $\varphi \colon \widetilde{W} \to W$ of D and an φ-exceptional normal crossing divisor $E = \sum_{i \in I} a_i E_i \subset \widetilde{W}$ such that $\varphi^{-1}(D)$ is a normal crossing divisor and such that $K_{\widetilde{W}} + \widetilde{D} = \varphi^*(K_W + D) + E$ with $a_i \geq 0$ (resp. $a_i \geq -1$) for all $i \in I$. If D is a reduced normal crossing divisor of W, then the pair (W, D) has only log-canonical singularities.

If X_0 is reduced and normal and has only canonical (resp. log-canonical) singularities, then the pair (X, X_0) has only canonical (resp. log-canonical) singularities by [27], [20, Th. 7.9] (resp. [20, Th. 7.5]). By [20, Cor. 11.13], the condition that X_0 has only canonical singularities is equivalent to the one that X_0 has only *rational* singularities since X_0 is reduced and normal. We refer to e.g. [20] and the references therein for more about related notions of singularities.

7.2. **Integration along fibers and (log-)canonical singularities.** Let $\varrho \in A_X^{n+1,n+1}$ and $\chi \in A_X^{n+1,n}$. Write $\varrho = \pi^*(ds \wedge d\bar{s}) \wedge R$ and $\chi = (\pi^*ds) \wedge K$, where $R \in C^\infty(X \setminus \operatorname{Sing} X_0, \Omega_{X/S}^n \wedge \overline{\Omega_{X/S}^n})$ and $K \in C^\infty(X \setminus \operatorname{Sing} X_0, \Omega_{X/S}^n \wedge \overline{\Omega_{X/S}^n})$. Define $\mathcal{R}(s) \in C^\infty(S^o)$ and $\mathcal{K}(s) \in C^\infty(S^o)$ as

$$\mathcal{R}(s) := \int_{X_s} R|_{X_s}, \qquad \mathcal{K}(s) := \int_{X_s} K|_{X_s},$$

so that $\pi_*(\varrho) = \mathcal{R}(s)\, ds \wedge d\bar{s}$ and $\pi_*(\chi) = \mathcal{K}(s)\, ds$.

Lemma 7.1. *Assume that X_0 is a reduced divisor of $\mathcal{X}$.*

(1) *If the pair $(\mathcal{X}, X_0)$ has only canonical singularities, then $\mathcal{R}(s), \mathcal{K}(s) \in \mathcal{B}(S)$ and $\mathcal{R}(0) = \int_{(X_0)_{\mathrm{reg}}} R|_{(X_0)_{\mathrm{reg}}}$, $\mathcal{K}(0) = \int_{(X_0)_{\mathrm{reg}}} K|_{(X_0)_{\mathrm{reg}}}$.*

(2) *If the pair $(\mathcal{X}, X_0)$ has only log-canonical singularities, then there exists $C > 0$ such that $|\mathcal{R}(s)| \leq C\,(-\log|s|)^n$.*

Proof. **Step 1** Since the pair $(\mathcal{X}, X_0)$ has only canonical (resp. log-canonical) singularities, there exist an embedded resolution $\varphi\colon \mathcal{Z} \to \mathcal{X}$ of X_0 and an φ-exceptional normal crossing divisor $E \subset \mathcal{Z}$ such that $K_{\mathcal{Z}} + \widetilde{X}_0 = \varphi^*(K_{\mathcal{X}} + X_0) + E$, $E = \sum_{i \in I} a_i D_i$. Here $\widetilde{X}_0 \subset \mathcal{Z}$ is the proper transform of X_0, D_i are irreducible and reduced divisors of $\mathcal{Z}$ and $a_i \geq 0$ (resp. $a_i \geq -1$) for all $i \in I$. We may assume that $Z_0 := \varphi^{-1}(X_0)$ is a normal crossing divisor of $\mathcal{Z}$ with $\widetilde{X}_0 \cup E \subset Z_0$. Set $\pi' := \pi \circ \varphi$. Then $Z_0 = (\pi')^{-1}(0)$ and $Z_s := (\pi')^{-1}(s) \cong X_s$ for $s \in S^o$.

Let $\mathcal{V} \subset X$ be an open subset, on which there is a nowhere vanishing holomorphic $n+1$-form Θ. By an argument using partition of unity, it suffices to prove the assertion when ϱ and χ are supported in $\mathcal{V}$. In what follows, we assume $\operatorname{supp} \varrho \subset \mathcal{V}$ and $\operatorname{supp} \chi \subset \mathcal{V}$.

Let $p \in Z_0 \cap \varphi^{-1}(\mathcal{V})$. Since X_0 is a reduced divisor of X, there is a coordinate neighborhood $(U, (z_0, \cdots, z_n))$ centered at p satisfying $\varphi(U) \subset \mathcal{V}$ and one of the following (a), (b): Write $\pi'|_U(z) = z_0^{e_0} \cdots z_n^{e_n}$, $e_i \in \mathbf{Z}_{\geq 0}$.

(a) If $p \in \widetilde{X}_0$, then $Z_s \cap U = \{z \in U;\ z_0 z_1^{e_1} \cdots z_n^{e_n} = s\}$ and $\widetilde{X}_0 \cap U = \{z_0 = 0\}$.

(b) If $p \notin \widetilde{X}_0$, then $Z_s \cap U = \{z \in U;\ z_0^{e_0} z_1^{e_1} \cdots z_n^{e_n} = s\}$, $e_0 > 0$ and $\widetilde{X}_0 \cap U = \emptyset$.

Since $e_0 = 1$ in case (a), we get the following expression on U:

(7.1)
$$(\pi')^*\left(\frac{ds}{s}\right) = \varphi^*\pi^*\left(\frac{ds}{s}\right) = \begin{cases} \frac{dz_0}{z_0} + \sum_{i \geq 1} e_i \frac{dz_i}{z_i} & \text{if} \quad p \in \widetilde{X}_0, \\ \sum_{i \geq 0} e_i \frac{dz_i}{z_i} & \text{if} \quad p \notin \widetilde{X}_0. \end{cases}$$

Step 2 Since $\Theta/\pi^*s \in H^0(\mathcal{V}, \omega_{\mathcal{X}}(X_0))$, $\varphi^*(\Theta/\pi^*s)$ is a meromorphic canonical form on $\mathcal{Z} \cap \varphi^{-1}(\mathcal{V})$ with at most logarithmic pole along $\widetilde{X}_0 \cap \varphi^{-1}(\mathcal{V})$ (resp. $\widetilde{Z}_0 \cap \varphi^{-1}(\mathcal{V})$) by the assumption $K_{\mathcal{Z}} + \widetilde{X}_0 = \varphi^*(K_{\mathcal{X}} + X_0) + \sum_{i \in I} a_i D_i$, $a_i \geq 0$ (resp. $a_i \geq -1$). Hence we get the following expression on U

(7.2)
$$\varphi^*\left(\frac{\Theta}{\pi^*s}\right) = \begin{cases} a(z) \frac{dz_0}{z_0} \wedge dz_1 \wedge \cdots \wedge dz_n & \text{if} \quad p \in \widetilde{X}_0 \cap U, \quad (\mathcal{X}, X_0) \text{ is C}, \\ b(z)\, dz_0 \wedge \cdots \wedge dz_n & \text{if} \quad p \notin \widetilde{X}_0 \cap U, \quad (\mathcal{X}, X_0) \text{ is C}, \\ a'(z) \frac{dz_0}{z_0} \wedge \cdots \wedge \frac{dz_n}{z_n} & \text{if} \quad p \in Z_0 \cap U, \quad (\mathcal{X}, X_0) \text{ is LC}, \end{cases}$$

where $a(z), b(z), a'(z) \in \mathcal{O}(U)$. Here we wrote C (resp. LC) for canonical (resp. log-canonical). Let $\Xi \in \Gamma(\mathcal{V} \setminus X_0, \Omega^n_{X/S})$ be such that $\Theta = \Xi \wedge \pi^* ds$. Since

$$(7.3) \qquad \varphi^*(\Theta/\pi^* s) = \varphi^* \Xi \wedge (\pi')^*(ds/s),$$

$\varphi^* \Xi$ is expressed as follows on $U \setminus Z_0$ by (7.1), (7.2), (7.3):

(7.4)
$$\varphi^* \Xi = \begin{cases} a(z)\, dz_1 \wedge \cdots \wedge dz_n \mod (\pi')^* ds & \text{if} \quad p \in \widetilde{X}_0 \cap U, \quad (\mathcal{X}, X_0) \text{ is C}, \\ \frac{z_0}{e_0} b(z)\, dz_1 \wedge \cdots \wedge dz_n \mod (\pi')^* ds & \text{if} \quad p \notin \widetilde{X}_0 \cap U, \quad (\mathcal{X}, X_0) \text{ is C}, \\ a'(z)\, \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n} \mod (\pi')^* ds & \text{if} \quad p \in Z_0 \cap U, \quad (\mathcal{X}, X_0) \text{ is LC}. \end{cases}$$

If $(\mathcal{X}, X_0)$ has only canonical singularities, then we set $c(z) := a(z)$ in case (a) and $c(z) := z_0 b(z)/e_0$ in case (b). If $(\mathcal{X}, X_0)$ has only log-canonical singularities, then we set $c(z) := a'(z)$. We always have $c(z) \in \mathcal{O}(U)$.

By (7.4), we get for all $F \in C_0^\infty(U)$ and $s \in S^o$

$$\int_{Z_s \cap U} F\, \varphi^*(\Xi \wedge \overline{\Xi})|_{Z_s \cap U} = \int_{Z_s \cap U} F(z)\, |c(z)|^2\, dz_1 \wedge \cdots \wedge dz_n \wedge \overline{dz_1 \wedge \cdots \wedge dz_n}.$$

Step 3　*In Steps 3,4, we assume that $(\mathcal{X}, X_0)$ has only canonical singularities.* Since the integrand is a C^∞ (n, n)-form on U, we get by [2, Th. 1]

(7.5)
$$\lim_{s \to 0} \int_{Z_s \cap U} F\, \varphi^*(\Xi \wedge \overline{\Xi}) = \int_{Z_0 \cap U} F(z)\, |c(z)|^2\, dz_1 \wedge \cdots \wedge dz_n \wedge \overline{dz_1 \wedge \cdots \wedge dz_n}$$
$$= \int_{\widetilde{X}_0 \cap U} F(z)\, |c(z)|^2\, dz_1 \wedge \cdots \wedge dz_n \wedge \overline{dz_1 \wedge \cdots \wedge dz_n}$$
$$= \int_{\widetilde{X}_0 \cap U} F\, \varphi^*(\Xi \wedge \overline{\Xi}).$$

Here we get the second equality as follows. In case (a), $(Z_0 \setminus \widetilde{X}_0) \cap U$ is defined locally by the equation $z_1^{e_1} \cdots z_n^{e_n} = 0$. Hence one of $dz_1, \ldots, dz_n$ vanishes on $(Z_0 \setminus \widetilde{X}_0) \cap U$, which implies the second equality of (7.5). In case (b), let $x \in (Z_0 \setminus \widetilde{X}_0) \cap U$. Then one of $z_0, \ldots, z_n$ vanishes on a neighborhood W of x in $(Z_0 \setminus \widetilde{X}_0) \cap U$. If $z_0|_W = 0$, then $c_i|_W = (z_0 b_i/e_0)|_W = 0$. If $z_j|_W = 0$ for some $j > 0$, then $dz_j|_W = 0$. Since $(Z_0 \setminus \widetilde{X}_0) \cap U$ is covered by these W, we get

$$\int_{(Z_0 \setminus \widetilde{X}_0) \cap U} F(z)\, |c(z)|^2\, dz_1 \wedge \cdots \wedge dz_n \wedge \overline{dz_1 \wedge \cdots \wedge dz_n} = 0.$$

Step 4　Let $\{U_\alpha\}_{\alpha \in A}$ be a covering of $\pi^{-1}(\mathcal{V})$ with $\#A < \infty$ such that on U_α, there is a system of coordinates satisfying (a) or (b) as in Step 2. Let $\{\chi_\alpha\}_{\alpha \in A}$ be a partition of unity subject to the covering $\{U_\alpha\}_{\alpha \in A}$.

Since $\varrho \in A_\mathcal{X}^{n+1,n+1}$ and $\chi \in A_\mathcal{X}^{n+1,n}$ have compact support in $\mathcal{V}$, there exist $m \in C^\infty(\mathcal{V})$ and $B \in A_\mathcal{X}^{0,q}$ with supp $B \subset \mathcal{V}$ such that

$$\varrho = (-1)^n m\, \Theta \wedge \overline{\Theta} = \pi^*(ds \wedge d\bar{s}) \wedge m\, \Xi \wedge \overline{\Xi}, \qquad \chi = (-1)^n \Theta \wedge B = (\pi^* ds) \wedge \Xi \wedge B.$$

Since $\pi_*\varrho = \pi_*(m\,\Xi\wedge\overline{\Xi})\,ds\wedge d\bar{s}$ and $\pi_*\chi = \pi_*(\Xi\wedge B)\,ds$, we get $\mathcal{R}(s) = \pi_*(m\,\Xi\wedge\overline{\Xi})$ and $\mathcal{K}(s) = \pi_*(\Xi\wedge B)$. Since

(7.6)

$$\lim_{s\to 0}\mathcal{R}(s) = \lim_{s\to 0}\int_{X_s}(m\,\Xi\wedge\overline{\Xi})|_{X_s} = \lim_{s\to 0}\sum_{\alpha\in A}\int_{Z_s\cap U_\alpha}\chi_\alpha\,\varphi^*(m\,\Xi\wedge\overline{\Xi})|_{Z_s\cap U_\alpha}$$

$$= \sum_{\alpha\in A}\int_{\widetilde{X}_0\cap U_\alpha}\chi_\alpha\,\varphi^*(m\,\Xi\wedge\overline{\Xi})|_{\widetilde{X}_0\cap U_\alpha} = \int_{\widetilde{X}_0}\varphi^*(m\,\Xi\wedge\overline{\Xi})|_{\widetilde{X}_0}$$

$$= \int_{(X_0)_{\mathrm{reg}}}m\,\Xi\wedge\overline{\Xi}|_{(X_0)_{\mathrm{reg}}} = \mathcal{R}(0)$$

by (7.5), we get $\mathcal{R}(s) \in C^0(S)$ by (7.6). Since $|s|^2\mathcal{R}(s) \in \mathcal{B}(S)$ by [31, Lemma 9.2], we get $\mathcal{R}(s) \in C^0(S)\cap|s|^{-2}\mathcal{B}(S) = \mathcal{B}(S)$.

Let $B \in A_{\mathcal{X}}^{0,n}$ with $\operatorname{supp}B \subset \mathcal{B}$. Since $B\wedge\pi^*d\bar{s} = h\,\overline{\Theta}$ with some $h \in C_0^\infty(\mathcal{V})$, we get $B = h\,\overline{\Xi}\mod\pi^*d\bar{s}$. Since $\Xi\wedge B = h\Xi\wedge\overline{\Xi}\mod\pi^*ds$ with $h \in C_0^\infty(\mathcal{V})$ and hence $\mathcal{K}(s) = \int_{X_s}h\Xi\wedge\overline{\Xi}|_{X_s}$, we get by (7.6)

(7.7)
$$\lim_{s\to 0}\mathcal{K}(s) = \int_{(X_0)_{\mathrm{reg}}}\Xi\wedge B|_{(X_0)_{\mathrm{reg}}} = \mathcal{K}(0)$$

and $\mathcal{K}(s) \in \mathcal{B}(S)$. This proves (1).

Step 5 Assume that $(\mathcal{X}, X_0)$ has only log-canonical singularities. By (7.4), there exists a constant $C > 0$ such that

$$\left|\int_{Z_s\cap U}F\,\varphi^*(\Xi\wedge\overline{\Xi})\right| \le C\left|\int_{z\in U,\,z_0^{e_1}\cdots z_n^{e_n}=s}\frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_n}{z_n}\wedge\overline{\frac{dz_1}{z_1}}\wedge\cdots\wedge\overline{\frac{dz_n}{z_n}}\right|$$

$$\le C\,(-\log|s|)^n.$$

This, together with the equality $\mathcal{R}(s) = \sum_{\alpha\in A}\int_{Z_s\cap U_\alpha}\chi_\alpha\,\varphi^*(m\,\Xi\wedge\overline{\Xi})|_{Z_s\cap U_\alpha}$, implies the desired estimate $|\mathcal{R}(s)| \le C\,(-\log|s|)^n$. This proves (2). $\qquad\square$

7.3. **The case of canonical singularities.** We assume that X_0 *is reduced and the pair* $(\mathcal{X}, X_0)$ *has only canonical singularities.* Set $l_q := h^q(X_s, \Omega_{X_s}^n(\xi_s^\vee))$.

Theorem 7.2. *The L^2 metric on $R^q\pi_*\omega_{X/S}(\xi)|_{S^\circ}$ extends to a continuous Hermitian metric of class $\mathcal{B}(S)$ on $R^q\pi_*\omega_{X/S}(\xi)|_S$ for all $q \ge 0$.*

Proof. After shrinking S if necessary, we get by Lemma 5.2 sections $\Psi_1,\dots,\Psi_{l_q} \in H^q(X,\Omega_X^{n+1}(\xi))$ such that $R^q\pi_*\Omega_X^{n+1}(\xi)|_S = \mathcal{O}_S\Psi_1\oplus\cdots\oplus\mathcal{O}_S\Psi_{l_q}$. We set

(7.8)
$$g_{\alpha\bar\beta}(s) := (\Psi_\alpha\otimes(\pi^*ds)^{-1}|_{X_s}, \Psi_\beta\otimes(\pi^*ds)^{-1}|_{X_s})_{L^2}$$

for $s \in S^\circ$. Then $g_{\alpha\bar\beta} \in C^\infty(S^\circ)$ and $\det(g_{\alpha\bar\beta}) > 0$ on S°. It suffices to prove that $g_{\alpha\bar\beta} \in \mathcal{B}(S)$ and $\det(g_{\alpha\bar\beta}) > 0$ on S.

There exist $\psi_\alpha \in H^0(X,\Omega_X^{n-q+1}(\xi))$ and $\Xi_\alpha \in H^0(X\setminus\operatorname{Sing}X_0,\Omega_{X/S}^{n-q}(\xi))$ by Theorem 5.1 such that

$$\Psi_\alpha = [\psi_\alpha\wedge\kappa_{\mathcal{X}}^q],\qquad \psi_\alpha|_{X\setminus\operatorname{Sing}X_0} = \Xi_\alpha\wedge\pi^*ds.$$

In $H^q(X\setminus X_0,\omega_{X/S}(\xi))$, we get the equality

(7.9)
$$\Psi_\alpha\otimes(\pi^*ds)^{-1} = [\Xi_\alpha\wedge\kappa_{\mathcal{X}}^q],\qquad \alpha = 1,\dots,l_q$$

under the canonical identification $\omega_{X/S}(\xi)|_{X \setminus X_0} = \Omega^n_{X/S}(\xi)|_{X \setminus X_0}$. Since Ξ_α is holomorphic and hence $\Xi_\alpha \wedge \kappa^q_\mathcal{X}|_{X_s}$ is the harmonic representative of the cohomology class $\Psi_\alpha \otimes (\pi^* ds)^{-1}|_{X_s}$ for $s \in S^o$, we deduce from (7.8), (7.9) that

$$(7.10) \qquad \pi_* (i^{(n-q+1)^2} h_\xi(\psi_\alpha \wedge \overline{\psi}_\beta) \wedge \kappa^q_\mathcal{X}) = i\, g_{\alpha\bar\beta}(s)\, ds \wedge d\bar{s}.$$

Since $\psi_\alpha \in H^0(X, \Omega^{n-q+1}_X(\xi))$, we see that $h_\xi(\psi_\alpha \wedge \overline{\psi}_\beta) \wedge \kappa^q_\mathcal{X}$ is a C^∞ $(n+1, n+1)$-form on X. By Lemma 7.1 (1) and (7.10), we get $g_{\alpha\bar\beta}(s) \in \mathcal{B}(S)$.

By (6.17) and the second inequality of Lemma 6.4, there exists $C > 0$ such that

$$(7.11)$$
$$\det \left(\Psi_\alpha \otimes (\pi^* ds)^{-1}|_{X_{\mu(t)}}, \Psi_\beta \otimes (\pi^* ds)^{-1}|_{X_{\mu(t)}} \right)_{L^2}$$
$$= |t|^{-2\delta^q_W} \det \left(\Theta_\alpha \otimes (f^* dt)^{-1}|_{Y_t}, \Theta_\beta \otimes (f^* dt)^{-1}|_{Y_t} \right)_{L^2} \geq C\, |t|^{-2\delta^q_W}.$$

Since $\delta^q_W \geq 0$, there exists $\epsilon > 0$ such that for all $s \in S^o$,

$$(7.12)$$
$$\det(g_{\alpha\bar\beta}(s)) = \det \left(\Psi_\alpha \otimes (\pi^* ds)^{-1}|_{X_s}, \Psi_\beta \otimes (\pi^* ds)^{-1}|_{X_s} \right)_{L^2} \geq \epsilon > 0.$$

Since $g_{\alpha\bar\beta}(s) \in \mathcal{B}(S)$, we get $\det(g_{\alpha\bar\beta}(0)) \geq \epsilon > 0$ by (7.12). $\qquad\square$

Corollary 7.3. *There exist $c \in \mathbf{R}$, $r \in \mathbf{Q}_{>0}$ and $\nu \in \mathbf{Z}_{\geq 0}$ such that as $s \to 0$*

$$\log \|\sigma^q_W(s)\|^2_{L^2} = c + O\left(|s|^r (\log|s|)^\nu \right).$$

Proof. Since $\mathrm{Hom}_G(W, R^q \pi_* \omega_{X/S}(\xi)) \otimes W \subset R^q \pi_* \omega_{X/S}(\xi)$ is a holomorphic sub-bundle, the result follows from Theorem 7.2. $\qquad\square$

Proof of Theorem 1.2 (2) By the definition of equivariant Quillen metrics,

$$(7.13)$$
$$\log \tau_G(X_s, \omega_{X_s}(\xi_s))(g)$$
$$= \log \|\sigma(s)\|^2_{\lambda_G(\omega_{X/S}(\xi)), Q}(g) - \sum_{q \geq 0,\, W \in \widehat{G}} (-1)^q \frac{\chi_W(g)}{\dim W} \log \|\sigma^q_W(s)\|^2_{L^2}.$$

The result follows from (7.13), Theorem 4.4 and Corollary 7.3. $\qquad\square$

Corollary 7.4. *If X_0 is reduced, normal and has only canonical singularities and if (ξ, h_ξ) is semi-negative in the dual Nakano sense on X, then there exist $c \in \mathbf{C}$, $r \in \mathbf{Q}_{>0}$, $l \in \mathbf{Z}_{\geq 0}$ such that as $s \to 0$*

$$\log \tau_G(X_s, \xi_s)(g) = (-1)^{n+1} \mathfrak{a}_g(X_0, \xi) \log |s|^2 + c + O\left(|s|^r (\log|s|)^l \right).$$

Proof. Since $(\xi^\vee, h_{\xi^\vee})$ is Nakano semi-positive on X, the result follows from Theorem 1.2 (2) and (6.23). $\qquad\square$

7.4. The case of log-canonical singularities. We assume that X_0 is reduced and the pair $(\mathcal{X}, X_0)$ has only log-canonical singularities. As before, we set $l_q = h^q(X_s, \Omega^n_{X_s}(\xi_s))$.

Lemma 7.5. *As $s \to 0$*

$$\log \det \left(\Psi_\alpha \otimes (\pi^* ds)^{-1}|_{X_s}, \Psi_\beta \otimes (\pi^* ds)^{-1}|_{X_s} \right)_{L^2} = O\left(\log(-\log|s|) \right).$$

In particular, as $s \to 0$,

$$\log \|\sigma^q_W(t)\|^2_{L^2} = O\left(\log(-\log|s|) \right).$$

Proof. Recall that $g_{\alpha\bar{\beta}}(s)$ was defined by (7.8). By (7.10) and the smoothness of the top form $h_\xi(\psi_\alpha \wedge \overline{\psi}_\beta) \wedge \kappa_{\mathcal{X}}^q$, we get the following estimate by Lemma 7.1 (2)

$$(7.14) \qquad |g_{\alpha\bar{\beta}}(s)| \leq C\,(-\log|s|^2)^n.$$

The result follows from (7.12) and (7.14). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 1.2 (1) By Proposition 6.8, Lemmas 6.9 and 7.5, we get $\mathbb{L}_g\left(\mu^*R\pi_*\omega_{X/S}(\xi)/Rf_*\omega_{Y/T}(F^*\xi)\right) = 0$. The result follows from Theorem 1.1. $\square$

8. Examples and questions

Example 8.1. Recall that β_g was defined in (6.22). We give examples of one-parameter families with $\alpha_g(X_0, \omega_{X/S}) \neq \beta_g$. Assume $G = \{1\}$ and write α, β for $\alpha_{\{1\}}(X_0, \omega_{X/S})$, $\beta_{\{1\}}$, respectively. We set $C := \mathbf{P}^1$. The inhomogeneous coordinate of C is denoted by $s = s_1/s_0$. For $d \in \mathbf{Z}_{>0}$, set

$$(8.1) \qquad \mathcal{X}' := \{([x], s) \in \mathbf{P}^{n+1} \times C;\ x_0^d + \cdots + x_n^d - s\,x_{n+1}^d = 0\}$$

and let $\mu\colon \mathcal{X} \to \mathcal{X}'$ be a projective resolution of the singularities of $\mathcal{X}'$. We set $\pi := \mathrm{pr}_2|_{\mathcal{X}'} \circ \mu$. Then the family $\pi\colon \mathcal{X} \to C$ is smooth over $C \setminus \{0, \infty\}$. Let $(\mathbf{C}^{n+1}, (z_0, \cdots, z_n))$ be the inhomogeneous coordinates of $\mathbf{P}^{n+1}$ defined by $z_i = x_i/x_{n+1}$. Since $\mathcal{X} \cap (\mathbf{C}^{n+1} \times \mathbf{C})$ is the graph of the holomorphic function $z_0^d + \cdots + z_n^d$ from $\mathbf{C}^{n+1}$ to $\mathbf{C}$, we may regard $\mathbf{C}^{n+1}$ as a coordinate neighborhood of $\mathcal{X}$ such that $\pi(z) = z_0^d + \cdots + z_n^d$. Then X_0 has a unique singular point at $\mathfrak{p} := ((0 : \cdots : 0 : 1), 0)$ and $\mathcal{O}_{X_0, \mathfrak{p}} \cong \mathbf{C}\{z_0, \ldots, z_n\}/(z_0^d + \cdots + z_n^d)$. We set $S := \{s \in C;\ |s| < 1\}$. Then $\Delta \cap S = \{0\}$. We compute α and β.

By (8.1), we regard $X_s \subset \mathbf{P}^{n+1}$ for $s \neq \infty$. Since the ideal sheaf $\mathcal{I}_{X_s}$ of X_s in $\mathbf{P}^{n+1}$ is isomorphic to $\mathcal{O}_{\mathbf{P}^{n+1}}(-d)$ and hence

$$\omega_{X_s} \cong \omega_{\mathbf{P}^{n+1}} \otimes N_{X_s/\mathbf{P}^{n+1}}|_{X_s} \cong \mathcal{O}_{\mathbf{P}^{n+1}}(d - n - 2)|_{X_s}$$

by the adjunction formula, we get for all $s \in S$ and $q \geq 0$

$$(8.2)$$

$$H^q(X_s, \omega_{X_s}) \cong \begin{cases} H^0(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(d - n - 2)) & (q = 0), \\ 0 & (0 < q < n), \\ H^{n+1}(\mathbf{P}^{n+1}, \Omega_{\mathbf{P}^{n+1}}) \cong \mathbf{C} & (q = n). \end{cases}$$

By (8.2), the L^2-metric on $R^q\pi_*\omega_{\mathcal{X}/C}|_S$ is smooth on S for $q > 0$, i.e.,

$$(8.3) \qquad \log\|\sigma^q(s)\|_{L^2} = O(1) \qquad (q > 0).$$

By the relative Serre duality, $\pi_*\omega_{\mathcal{X}/C} \cong (\mathrm{pr}_2)_*\omega_{(\mathbf{P}^{n+1} \times C)/C}(\mathcal{X})$, where the isomorphism is induced by the long exact sequence of direct images associated to the following short exact sequence of sheaves on $\mathbf{P}^{n+1} \times C$

$$0 \longrightarrow \omega_{(\mathbf{P}^{n+1} \times C)/C} \longrightarrow \omega_{(\mathbf{P}^{n+1} \times C)/C}(\mathcal{X}) \longrightarrow \omega_{\mathcal{X}/C} \longrightarrow 0$$

and by the vanishing $R^q\pi_*\omega_{(\mathbf{P}^{n+1} \times C)/C} = 0$ for $q < n + 1$.

Set $z^e := z_0^{e_0} \cdots z_n^{e_n}$ and $|e| = \sum e_i$. A basis of $H^0(\mathbf{P}^{n+1}, \Omega_{\mathbf{P}^{n+1}}^{n+1}(X_s)) \cong H^0(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(d-n-2))$ is given by $\{z^e dz_0 \wedge \cdots \wedge dz_n/(z_0^d + \cdots + z_n^d - s)\}_{|e| \leq d-(n+2)}$ and the corresponding basis of $\pi_*\omega_{\mathcal{X}/C}|_S$ as a free $\mathcal{O}_S$-module is given by

$$\left\{\omega^e(s) := \frac{z^e}{z_0^{d-1}} dz_1 \wedge \cdots \wedge dz_n|_{X_s}\right\}_{|e| \leq d-(n+2)}.$$

We put $\sigma^0(s) := \bigwedge_{|e| \leq d-(n+2)} \omega^e(s) \in \det H^0(X_s, \Omega_{X_s}^n)$. Then σ^0 gives a basis of the rank one $\mathcal{O}_S$-module $\det \pi_* \omega_{X/S}$. We define

$$\psi_s \colon \mathbf{P}^{n+1} \ni (z_0 : \cdots : z_{n+1}) \to (s^{\frac{1}{d}} z_0 : \cdots : s^{\frac{1}{d}} z_n : z_{n+1}) \in \mathbf{P}^{n+1}.$$

Since $\psi_s \colon X_1 \to X_s$ is an isomorphism with $\psi_s^* \omega^e(s) = s^{\frac{n+|e|+1-d}{d}} \omega^e(1)$, we get

$$(8.4) \qquad \psi^* \sigma^0(s) = s^{\sum_{|e| \leq d-(n+2)} (n+|e|+1-d)/d} \sigma^0(1) = s^{-\binom{d}{n+2}/d} \sigma^0(1).$$

Since

$$\|\sigma^0(s)\|_{L^2(X_s)}^2 = \det\left(\int_{X_s} i^{n^2} \omega^e(s) \wedge \overline{\omega^{e'}(s)}\right)_{|e|,|e'| \leq d-(n+2)}$$

$$= \det\left(\int_{X_1} i^{n^2} \psi_s^* \omega^e(s) \wedge \overline{\psi_s^* \omega^{e'}(s)}\right)_{|e|,|e'| \leq d-(n+2)} = \|\psi_s^* \sigma^0(s)\|_{L^2(X_1)}^2,$$

we get by (8.4)

$$(8.5) \qquad \log \|\sigma^0(s)\|_{L^2}^2 = -\frac{1}{d}\binom{d}{n+2} \log |s|^2 + \log \|\sigma^0(1)\|_{L^2}^2.$$

Substituting (8.3) and (8.5) into the first equality of (6.20), we get

$$(8.6) \qquad \beta = \alpha + \frac{1}{d}\binom{d}{n+2}.$$

By [31, Th. 8.1] and (4.15), we get

$$(8.7) \qquad \alpha = (-1)^{n+1} \frac{(-1)^n}{(n+2)!} \mu(X_0, \mathfrak{p}) = -\frac{(d-1)^{n+1}}{(n+2)!},$$

where $\mu(X_0, \mathfrak{p})$ is the Milnor number of $(X_0, \mathfrak{p})$. By (8.6), (8.7), we get

$$(8.8) \qquad \beta = \frac{1}{d}\binom{d}{n+2} - \frac{(d-1)^{n+1}}{(n+2)!}.$$

In particular, if $d \geq n+2$, we get $\alpha \neq \beta$ in this example.

Let us compute β by using Lemma 6.9. If $d \leq n+1$, then $\delta^0 = 0$ by (8.2). Assume $d \geq n+2$. Let $r \colon \widetilde{X} \to X$ be the blowing-up at $\mathfrak{o}$ and set $\widetilde{\pi} := r \circ \pi \colon \widetilde{X} \to S$. Then $\widetilde{\pi}^{-1}(0) = \widetilde{X}_0 + dE$, where $\widetilde{X}_0$ is the proper transform of X_0 and $E = r^{-1}(\mathfrak{o})$ is the exceptional divisor intersecting $\widetilde{X}_0$ transversally. Define $\mu \colon T \to S$ by $\mu(t) = t^d$. Then the singular locus of $\widetilde{X} \times_S T$ is locally isomorphic to the product of the isolated two-dimensional singularity $\{(x, y, s) \in \mathbf{C}^3; xy^d = s^d\}$ and $\mathbf{C}^{n-1}$. Let $p \colon Y \to \widetilde{X} \times_S T$ be the minimal resolution and set $f := \mathrm{pr}_2 \circ p$. Then $f \colon (Y, Y_0) \to (T, 0)$ is a semistable reduction of $\pi \colon (X, X_0) \to (S, 0)$. We set $F := r \circ \mathrm{pr}_1 \circ p$. Since $\|\omega^e(s)\|_{L^2}^2 = |s|^{2(n+|e|+1-d)} \|\omega^e(1)\|_{L^2}^2$ by the relations $\psi_s^* \omega^e(s) = s^{\frac{n+|e|+1-d}{d}} \omega^e(1)$ and $s = t^d$, $t^k \mu^* \omega^e$ is an element of $\Gamma(T, f_* \omega_{Y/T})$ if and only if $k \geq d - (n+1) - |e|$ by Proposition 6.6. Hence

$$(8.9) \qquad \mu^*(\pi_* \omega_{X/S})/f_* \omega_{Y/T} \cong \bigoplus_{|e| \leq d-(n+2)} \mathcal{O}_T/\mathfrak{m}_0^{d-(n+1)-|e|}.$$

By (8.9), $\delta^0 = \binom{d}{n+2}$. Since $\delta^q = 0$ for $q > 0$ by (8.2), we get (8.6) by Lemma 6.9.

Let $p \colon \widetilde{X}_0 \to X_0$ be the blowing-up at $\mathfrak{p}$ and set $E := p^{-1}(\mathfrak{p})$. By [20, (3.8.1)], $K_{\widetilde{X}_0} = p^* K_{X_0} + (n-d)E$. Since the isolated singularity $(X_0, \mathfrak{p})$ is log-canonical if and only if $d \leq n+1$, Eq.(8.6) is compatible with Theorem 1.2.

Question 8.2. If all the direct images $R^q\pi_*\omega_{X/S}(\xi)$, $q \geq 0$ are locally free on S, does Theorem 1.1 remain valid without assuming the Nakano semi-positivity of ξ? In general, what can one say about the singularity of $\log \tau_G(X_s, \omega_{X_s}(\xi_s))(g)$ without the assumption of the Nakano semi-positivity of ξ? Does the logarithmic divergence still hold? If it is the case, is the coefficient of $\log |s|^2$ expressed by the topological term $\alpha_g(X_0, \omega_{X/S}(\xi))$ and the g-action on the torsion modules $\ker(\varphi)$ and $\mathrm{coker}(\varphi)$, where $\varphi\colon R^q f_*\omega_{Y/T}(F^*\xi) \to \mu^* R^q\pi_*\omega_{X/S}(\xi)$ is the natural map?

Question 8.3. If $\mu^* R^q\pi_*\omega_{X/S}(\xi) = R^q f_*\omega_{Y/T}(F^*\xi)$ for all Nakano semi-positive vector bundles on X and for all $q \geq 0$, then is the pair $(\mathcal{X}, X_0)$ log-canonical?

Question 8.4. Is the Lefschetz trace $\mathbb{L}_g(\mu^* R\pi_*\omega_{X/S}(\xi)/Rf_*\omega_{Y/T}(F^*\xi))$ expressed by some other geometric data like the discrepancies of $(\mathcal{X}, X_0)$, the monodromy of the family $\pi\colon X \to S$ etc.?

Question 8.5. Are the coefficients ν_g, c_g in Theorem 1.1 expressible in terms of some local or global geometric data associated to the family $\pi\colon (X, X_0) \to (S, 0)$?

9. Appendix

In this section, we prove some technical results. We keep the notation in Sect. 6.

9.1. **A sequence of Kähler forms on Z approximating κ_Z.** The construction of a sequence of Kähler forms $\{\kappa_{Z,k}\}$ in the proof of Proposition 6.2 is as follows. (Although the construction of such a sequence of Kähler forms is standard, we give it here for the completeness reason and its use in Sect. 9.2.) Since $X \times_T S$ is a hypersurface of a complex manifold $X \times S$ and since $\rho\colon Z \to X \times_S T$ is a composite of blowing-ups with non-singular centers, there is a sequence by e.g. [1, Sect. 1.2], [4, Th. 13.2]

$$
\begin{array}{ccccccccc}
W & \to & W_{m-1} & \to \cdots \to & W_1 & \to & X \times S \\
\cup & & \cup & & \cup & & \cup \\
Z & \to & Z_{m-1} & \to \cdots \to & Z_1 & \to & X \times_T S
\end{array}
$$

where all W_i are smooth projective algebraic manifolds, each morphism $\pi_i\colon W_{i+1} \to W_i$ is given by the blowing-up with non-singular center $C_i \subset W_i$, and Z_{i+1} is the proper transform of Z_i under the blowing-up $\pi_i\colon W_{i+1} \to W_i$. We set $\rho_i := \pi_i|_{Z_{i+1}}\colon Z_{i+1} \to Z_i$. Then $\rho = \rho_0 \circ \cdots \circ \rho_{m-1}$. Let E_i be the exceptional divisor of $\pi_i\colon W_{i+1} \to W_i$. Then $E_i \cong \mathbf{P}(N_{C_i/W_i})$, where N_{C_i/W_i} is the normal bundle of C_i in N_i. Let $\mathcal{O}_{W_i}(E_i)$ be the line bundle on W_i defined by the effective divisor E_i and let σ_i be the canonical section of $\mathcal{O}_{W_i}(E_i)$ such that $E_i = \mathrm{div}(\sigma_i)$.

Since $\mathrm{pr}_2 \circ \pi_0 \circ \cdots \circ \pi_i(E_i) = \{0\}$, there is a small open neighborhood $\mathcal{V}_i^{(k)} \subset W_{i+1}$ of E_i such that $\mathrm{pr}_2 \circ \pi_0 \circ \cdots \circ \pi_i(\mathcal{V}_i^{(k)}) \subset T(\frac{1}{k})$. Since $\mathcal{O}_{W_i}(E_i)|_{E_i} \cong \mathcal{O}_{\mathbf{P}(N_{C_i/W_i})}(-1)$, there is a Hermitian metric $h_{\mathcal{O}_{W_i}(E_i)}$ on $\mathcal{O}_{W_i}(E_i)$ such that $h_{\mathcal{O}_{W_i}(E_i)}(\sigma_i, \sigma_i) = 1$ on $W_i \setminus \mathcal{V}_i$ and such that $c_1(\mathcal{O}_{W_i}(-E_i), h^{-1}_{\mathcal{O}_{W_i}(E_i)})|_{E_i} = dd^c \log h_{\mathcal{O}_{W_i}(E_i)}(\sigma_i, \sigma_i)|_{E_i}$ is a positive $(1,1)$-form on the relative tangent bundle $T\mathbf{P}(N_{C_i/W_i})/C_i$. For a Kähler form ω_{W_i} on W_i, there exists $A_i > 0$ such that $A_i \pi_i^*\omega_{W_i} + c_1(\mathcal{O}_{W_i}(-E_i), h^{-1}_{\mathcal{O}_{W_i}(E_i)})$ is a positive $(1,1)$-form on $TW_i|_{E_i}$ and hence on a neighborhood of E_i. Choosing A_i large enough, we may assume that $A_i \pi_i^*\omega_{W_i} + c_1(\mathcal{O}_{W_i}(-E_i), h^{-1}_{\mathcal{O}_{W_i}(E_i)})$ is a positive $(1,1)$-form on $\mathcal{V}_i^{(k)}$. Since $A_i \pi_i^*\omega_{W_i} + c_1(\mathcal{O}_{W_i}(-E_i), h^{-1}_{\mathcal{O}_{W_i}(E_i)}) = A_i \pi_i^*\omega_{W_i}$ on $W_i \setminus \mathcal{V}_i^{(k)}$, $\widetilde{\omega}_{W_i}^{(k)} := \pi_i^*\omega_{W_i} + \frac{1}{A_i} c_1(\mathcal{O}_{W_i}(-E_i), h^{-1}_{\mathcal{O}_{W_i}(E_i)})$ is a Kähler form on W_i

such that $\widetilde{\omega}_{W_i}^{(k)} = \pi_i^* \omega_{W_i}$ on $W_i \setminus \mathcal{V}_i^{(k)}$ and $[\widetilde{\omega}_{W_i}^{(k)}] = \pi_i^* [\omega_{W_i}] + \frac{1}{A_i} c_1(\mathcal{O}_{W_i}(-E_i))$. (See e.g. [12, Prop. 12.4] for more details.)

We set $\omega_{W_0} := \kappa_X + \kappa_S$. For $k > 1$, we get a sequence of Kähler forms $\{\omega_{W_i}^{(k)}\}$ by the procedure as above. We define $\kappa_{Z,k} := \omega_W^{(k)}|_Z$. Write $\rho^{-1}(\Sigma) = \bigcup_{\lambda \in \Lambda} D_\lambda$. By the construction of $\omega_W^{(k)}$, there is a Hermitian metric $h_{\mathcal{O}_Z(D_\lambda),k}$ on $\mathcal{O}_Z(D_\lambda)$ and a real number $a_\lambda^{(k)} \geq 0$ for every $\lambda \in \Lambda$ such that

$$(9.1) \qquad \kappa_{Z,k} = \kappa_Z - \sum_{\lambda \in \Lambda} a_\lambda^{(k)} c_1(\mathcal{O}_Z(D_\lambda), h_{\mathcal{O}_Z(D_\lambda),k})$$

is a Kähler form on Z with $\kappa_{Z,k} = \kappa_Z$ on $Z \setminus \varpi^{-1}(T(\frac{1}{k}))$. Here $a_\lambda^{(k)} = 0$ if D_λ is not an exceptional divisor of $\rho \colon Z \to X \times_T S$ by construction. By (9.1), we get

$$(9.2) \qquad [\kappa_{Z,k}]|_{Z \setminus Z_0} = [\kappa_Z]|_{Z \setminus Z_0},$$

since $\mathcal{O}_Z(D_\lambda)|_{Z \setminus Z_0} \cong \mathcal{O}_Z|_{Z \setminus Z_0}$ for all $\lambda \in \Lambda$. By (6.2), (9.2), we again get (6.4).

9.2. Takegoshi's theorem with respect to the degenerate Kähler form.
Since $\rho \colon Z \to X \times_T S$ is a resolution obtained as a composite of blowing-ups with non-singular centers, there is an extension of Takegoshi's theorem (Theorem 5.1) for Z by Mourougane-Takayama [23, Prop. 4.4] with respect to the degenerate Kähler form $\kappa_Z = \rho^*(\kappa_X + \kappa_S)$. However, it is not immediate from their proof if the operator L^q in [23, Prop. 4.4] is given by the multiplication by κ_Z^q. We make this point clear by proving the following:

Proposition 9.1. *For $u \in H^q(Z, \Omega_Z^{n+1}(\rho^*\xi))$, there exists $\sigma \in H^0(Z, \Omega_Z^{n+1-q}(\rho^*\xi))$ such that $u = [\sigma \wedge \kappa_Z^q]$ and $(\varpi^* dt) \wedge \sigma = 0$.*

Proof. We write $Z_0 = \bigcup_{\lambda \in \Lambda} D_\lambda$. By the construction in Sect. 9.1, there is a Kähler form ω_Z on Z of the form $\omega_Z = \kappa_Z - \sum_{\lambda \in \Lambda} a_\lambda c_1(\mathcal{O}_Z(D_\lambda), h_{\mathcal{O}_Z(D_\lambda)})$, where $a_\lambda > 0$ is a constant. By Theorem 5.1, there exists $\sigma \in H^0(Z, \Omega_Z^{n+1-q}(\rho^*\xi))$ such that $u = [\sigma \wedge \omega_Z^q]$ and $(\varpi^* ds) \wedge \sigma = 0$. Since $\sigma \wedge (\omega_Z^q - \kappa_Z^q) = \sigma \wedge (\omega_Z - \kappa_Z) \wedge \sum_{i+j=q-1} \omega_Z^i \kappa_Z^j$, it suffices to prove the equality of cohomology classes on Z

$$[\sigma \wedge (\omega_Z - \kappa_Z)] = 0.$$

Let s_λ be the canonical section of $\mathcal{O}_Z(D_\lambda)$ such that $\mathrm{div}(s_\lambda) = D_\lambda$. Since

$$\sigma \wedge (\omega_Z - \kappa_Z) = \bar{\partial}\{\sigma \wedge \sum_{\lambda \in \Lambda} a_\lambda \, \partial \log h_{\mathcal{O}_Z(D_\lambda)}(s_\lambda, s_\lambda)/2\pi i\},$$

it suffices to prove that $\sigma \wedge \partial \log h_{\mathcal{O}_Z(D_\lambda)}(s_\lambda, s_\lambda)$ is a C^∞ differential form on Z.

Set $\mathfrak{D} := \bigcup_{\lambda \neq \lambda'} D_\lambda \cap D_{\lambda'}$ and $D_\lambda^o := D_\lambda \setminus \mathfrak{D}$. Let $p \in D_\lambda^o$. There is a system of coordinates $(\mathcal{V}, (z_0, \dots, z_n))$ of Z centered at p such that $\varpi(z) = \epsilon(z) z_0^{e_\lambda}$ for some $e_\lambda \in \mathbf{Z}_{>0}$ and a nowhere vanishing $\epsilon(z) \in \mathcal{O}(\mathcal{V})$. Since $\varpi(z) = \epsilon(z) z_0^{e_\lambda}$ and hence $\varpi^* ds = e_\lambda \epsilon(z) z_0^{e_\lambda - 1} dz_0 + z_0^{e_\lambda} d\epsilon$, the condition $(\varpi^* ds) \wedge \sigma = 0$ implies that

$$(9.3) \qquad \frac{dz_0}{z_0} \wedge \sigma = -\frac{1}{e_\lambda \epsilon(z)} d\epsilon(z) \wedge \sigma \in \Omega_{\mathcal{V}}^{n+2-q}(\rho^*\xi).$$

Let $p \in \mathfrak{D}$. There is a system of coordinates $(\mathcal{U}, (z_0, \dots, z_n))$ of Z centered at p such that $\varpi(z) = \epsilon(z) z_0^{e_0} \cdots z_n^{e_n}$ for some $e_i \in \mathbf{Z}_{>0}$ and a nowhere vanishing

$\epsilon(z) \in \mathcal{O}(\mathcal{U})$. If $e_i > 0$, then we deduce from (9.3) that $(\frac{dz_i}{z_i} \wedge \sigma)|_{\mathcal{U} \setminus \mathfrak{D}} \in \Omega^{n+2-q}_{\mathcal{U} \setminus \mathfrak{D}}(\rho^*\xi)$. Since $\mathfrak{D}$ has codimension 2 in $\mathcal{U}$, we get by the Hartogs extension theorem

$$(9.4) \qquad \frac{dz_i}{z_i} \wedge \sigma \in \Omega^{n+2-q}_{\mathcal{U}}(\rho^*\xi).$$

Let $p \in Z_0$. There is a system of coordinates $(\mathcal{U}, (z_0, \dots, z_n))$ as above such that $\varpi|_{\mathcal{U}}(z) = \epsilon(z)\, z_0^{e_0} \cdots z_n^{e_n}$. There exist $\phi \in C^\infty(\mathcal{U})$ and $\nu_i \in \mathbf{R}$ such that

$$(9.5) \qquad \log h_{\mathcal{O}_Z(D_\lambda)}(s_\lambda, s_\lambda)|_{\mathcal{U}} = \sum_{i=0}^n \nu_i \log |z_i|^2 + \phi.$$

Here $e_i = 0$ implies $\nu_i = 0$. By (9.4) and (9.5), we get

(9.6)

$$\sigma \wedge \partial \log h_{\mathcal{O}_Z(D_\lambda)}(\sigma_\lambda, \sigma_\lambda)|_U = \sum_{i=0}^n \nu_i\, \sigma \wedge \frac{dz_i}{z_i} + \sigma \wedge \partial\phi \in A^{n+2-q,0}_{\mathcal{U}}(\rho^*\xi).$$

Since $p \in Z_0$ is an arbitrary point, we get $\sigma \wedge \partial \log h_{\mathcal{O}_Z(D_\lambda)}(\sigma_\lambda, \sigma_\lambda) \in A^{n+2-q,0}_Z(\rho^*\xi)$. This completes the proof. $\qquad\square$

As a consequence of Proposition 9.2, we get Takegoshi's theorem (Theorem 5.1) for $(Y, F^*\xi)$ with respect to the degenerate Kähler form $\kappa_Y = r^*(\kappa_X + \kappa_T)$.

Proposition 9.2. *For $v \in H^q(Y, \Omega^{n+1}_Y(F^*\xi))$, there exists $\tau \in H^0(Y, \Omega^{n+1-q}_Y(F^*\xi))$ such that $v = [\tau \wedge \kappa_Y^q]$ and $(f^*dt) \wedge \tau = 0$.*

Proof. Set $u := q^*v \in H^q(Z, \Omega^{n+1}_Z(\rho^*\xi))$. By Proposition 9.1, there exists $\sigma \in H^0(Z, \Omega^{n+1-q}_Z(\rho^*\xi))$ such that $u = [\sigma \wedge \kappa_Z^q]$ and $(\varpi^*ds) \wedge \sigma = 0$. By the proof of Proposition 6.2, there exists $\tau \in H^0(Y, \Omega^{n+1-q}_Y(F^*\xi))$ such that $q^*\tau = \sigma$. Since $q^*\kappa_Y = \kappa_Z$, we get $q^*(v - [\tau \wedge \kappa_Z^q]) = 0$. Since the map of cohomologies $q^*\colon H^q(Y, \Omega^{n+1}_Y(F^*\xi)) \to H^q(Z, \Omega^{n+1}_Z(\rho^*\xi))$ is an isomorphism by [28, Th. 6.9 (i)], we get $v = [\tau \wedge \kappa_Y^q]$. We get the equality $(\varpi^*dt) \wedge \sigma = 0$ as before in the proof of Proposition 6.2. $\qquad\square$

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